Nonmaximal weak-*Dirichlet algebras

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0. Introduction

Let $A$ be a weak-*Dirichlet algebra of $L^\infty(m)$ which was introduced by Srinivasan and Wang [7]. Let $H^\infty(m)$ denote the weak-*closure of $A$ in $L^\infty(m)$. Suppose there exists at least one positive nonconstant function $\nu$ in $L^1(m)$ such that the measure $\nu dm$ is multiplicative on $A$. Then Merrill [4] characterizes the classical space $H^\infty(d\theta)$ by invariant subspaces of $H^\infty(m)$ or the maximality of $H^\infty(m)$ as a weak-*closed subalgebra of $L^\infty(m)$. In section 1 we characterize $H^\infty(d\theta d\phi)$, which is certain weak-* Dirichlet algebra on the torus, by invariant subspaces of $H^\infty(m)$. We need not assume the existence of the above $\nu$. Then, in some special case, Muhly [6] shows that $H^\infty(m)$ is a maximal weak-*closed subalgebra of $L^\infty(m)$. But in general, $H^\infty(m)$ is not maximal and so there exist weak-* closed subalgebras of $L^\infty(m)$ which contain $H^\infty(m)$ properly. In section 2 we construct some typical subalgebra in such subalgebras and we determine forms of all weak-*closed subalgebras which contain this subalgebra. This is applied to determine forms of all subalgebras which contain $H^\infty(d\theta d\phi)$.

Recall that by definition a weak-*Dirichlet algebra is an algebra $A$ of essentially bounded measurable functions on a probability measure space $(X, \mathcal{M}, m)$ such that (i) the constant functions lie in $A$; (ii) $A + \overline{A}$ is weak-*dense in $L^\infty(m)$ (the bar denotes conjugation, here and always); (iii) for all $f$ and $g$ in $A$, $\int f g dm = (\int f dm)(\int g dm)$. The abstract Hardy spaces $H^p(m)$, $1 \leq p \leq \infty$, associated with $A$ are defined as follows. For $1 \leq p < \infty$, $H^p(m)$ is the $L^p(m)$-closure of $A$, while $H^\infty(m)$ is defined to the weak-*closure of $A$ in $L^\infty(m)$. For $1 \leq p \leq \infty$, $H^p_0 = \{f \in H^p(m) : \int f dm = 0\}$. For any subset $M \subseteq L^\infty(m)$, denote by $[M]_2$ the $L^2(m)$-closure of $M$. A closed subspace $M$ of $L^p(m)$ is called $B$-invariant if $f \in M$ and $g \in B$ imply that $fg \in M$, where $B$ is a subalgebra of $L^\infty(m)$. In particular, if $B = L^\infty(m)$, $M$ is called doubly-invariant. For any measurable subset $E$ of $X$, the function $\chi_E$ is the characteristic function of $E$. If $f \in L^p(m)$, write $E_f$ for the support set of $f$ and write $\chi_f$ for the characteristic function of $E_f$.

We use the following result.

(a) If $M$ is a weak-*closed $A$-invariant subspace of $L^\infty(m)$, then $M=$
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For weak-*Dirichlet algebras this has never published, but the proof is easy if we use the logmodularity of $H^\infty(m)$.

1. Characterization of $H^\infty(d\theta d\phi)$

Let $A$ be the algebra of continuous, complex-valued functions on the torus $T^2 = \{(z, w) : |z| = |w| = 1\}$ which are uniform limits of polynomials in $z^n w^m$ where

$$(n, m) \in \Gamma = \{(n, m) : m > 0\} \cup \{(n, 0) : n \geq 0\}.$$

Denoting the normalized Haar measure on $T^2$ by $d\theta d\phi$, then $A$ is a weak-*Dirichlet algebra of $L^\infty(d\theta d\phi)$. Recall $H^\infty(d\theta d\phi)$ is the weak-*closure of $A$ in $L^\infty(d\theta d\phi)$.

In general, let $A$ be a weak-*Dirichlet algebra of $L^\infty(m)$. Suppose there exists at least one positive nonconstant function $v$ in $L^1(m)$ such that for all $f$ and $g$ in $A$, $\int f g v \, dm = (\int f \, dm)(\int g v \, dm)$. Then by the logmodularity of $H^\infty(m)$, $H^\infty_0 = ZH^\infty(m)$ for some inner function $Z$ in $H^\infty(m)$, where a function $f \in H^\infty(m)$ is called inner if $|f| = 1$ a.e.

In [4] Merrill obtains the following result for the characterization of the classical space $H^\infty(d\theta)$.

(b) The following properties for $H^\infty(m)$ are equivalent.
(1) $H^\infty(m)$ is isomorphic to the classical space $H^\infty(d\theta)$.
(2) Every nonzero weak-*closed $A$-invariant subspace of $H^\infty(m)$ has the form

$$M = FH^\infty(m)$$

where $F$ is an inner function in $M$.

(3) $H^\infty(m)$ is a maximal weak-*closed subalgebra of $L^\infty(m)$.

In this section we characterize $H^\infty(d\theta d\phi)$ which is not a maximal weak-*Dirichlet algebra [4]. Let $J^\infty$ be the weak-*closure of $\bigcup_{n=0}^\infty Z^n H^\infty(m)$ and let $I^\infty$ be $\bigcap_{n=0}^\infty Z^n H^\infty_0$.

Theorem 1. (1) $J^\infty$ is the minimum weak-*closed subalgebra of $L^\infty(m)$ which contains $H^\infty(m)$ properly. (2) $I^\infty$ is the maximal weak-*closed ideal of $J^\infty$ in $H^\infty(m)$.

Proof. First, we shall show that if $B$ is a weak-*closed subalgebra of $L^\infty(m)$ such that $B \supseteq H^\infty(m)$, then $B \supseteq J^\infty$. If $m$ is multiplicative on $B$, then $\overline{B}$ is orthogonal to $H^\infty_0$ and hence $B \subseteq H^2(m)$ [7, p 226] and hence $B \subseteq H^2(m) \cap L^\infty(m) = H^\infty(m)$ by (a) in Introduction. This contradicts to
$B \supseteq H^\omega(m)$. If $m$ is not multiplicative on $B$, the function $Z$ has the inverse in $B$. For, if not, there exists a complex homomorphism $\phi$ on $B$ such that $\phi(Z) = 0$. Then $\ker \phi \supseteq H^\omega = ZH^\omega(m)$. If $\phi$ is restricted to $H^\omega(m)$, then $\ker \phi = H^\omega_0$, so by the logmodularity, the unique representing measure of $\phi$ is $m$. This contradicts that $m$ is not multiplicative on $B$. Thus $B$ is the weak-*closed subalgebra of $L^\omega(m)$ that contains $\overline{Z}$ and $H^\omega(m)$, so $B \supseteq J^\omega$. This proves (1).

Now if $K$ is the weak-*closed ideal of $J^\omega$ such that $I^\omega \subseteq K \subseteq H^\omega(m)$, since both $Z$ and $\overline{Z}$ is in $J^\omega$, the subalgebra $K = ZK$. Thus $K \subseteq \bigcap_{n=1}^\infty Z_n H^\omega(m)$ = $I^\omega$. It is known [5] that $I^\omega$ is the ideal of $J^\omega$. This proves (2).

Denote by $\mathcal{A}^p$ ($1 \leq p \leq \infty$) the closure in $L^p(m)$ (weak-*closure for $p = \infty$) of polynomials in $Z$. Denote by $\mathcal{H}^p$ ($1 \leq p \leq \infty$) the closure in $L^p(m)$ (weak-* closure for $p = \infty$) of polynomials in $Z$ and $\overline{Z}$. Let $I^p$ be the closure of $I^\omega$ in $L^p(m)$ and let $\mathcal{S}^p$ be the closure of $I^p + \overline{I}^p$ in $L^p(m)$ and let $J^p$ be the closure of $J^\omega$ in $L^p(m)$. The following result is known [4, Lemma 5].

(c) If $1 \leq p \leq \infty$, then

\[ H^p(m) = \mathcal{A}^p + I^p, \quad L^p(m) = \mathcal{H}^p + \mathcal{S}^p, \]

\[ J^p = \mathcal{S}^p + I^p \]

where $+$ denotes algebraic direct sum and if $p = 2$, each decomposition is orthogonal.

If $1 < p < \infty$, we can show easily that $L^p(m) = J^p + \overline{I}^p$.

The following result is known, too [3].

(d) For $1 \leq p \leq \infty$, there exists an isometric-*isomorphism (i.e., taking complex conjugates into complex conjugates) between $L^p(d\theta)$ of the disc and $\mathcal{H}^p$ in $L^p(m)$, which maps the classical space $H^p(d\theta)$ onto $\mathcal{A}^p$ in $H^p(m)$.

We can prove the following results (e) and (f). The proofs are almost parallel to those of (c) and (d). Suppose there exists a nontrivial inner function $W$ in $I^\omega$. Denote by $H^p$ ($1 \leq p \leq \infty$) the closure in $L^p(m)$ (weak-* closure for $p = \infty$) of polynomials in $Z_n W^m$ where $(n, m) \in \Gamma$. Then $H^p$ is a subspace (subalgebra for $p = \infty$) of $H^\omega(m)$ by $ZI^p = I^p$ which (2) in theorem 1 shows. Denote by $L^p$ ($1 < p < \infty$) the closure in $L^p(dm)$ (weak-* closure for $p = \infty$) of polynomials in $Z$, $\overline{Z}$, $W$ and $\overline{W}$. Let

\[ S^p = \{ f \in H^p(m) : \int Z^m W^m f dm = 0, \ (n, m) \in \Gamma \} . \]

Denote by $\mathcal{S}^p$ the closure of $S^p + S^p$ in $L^p(m)$ (weak-*closure for $p = \infty$).

(e) For $1 \leq p \leq \infty$, there exists an isometric-*isomorphism between $L^p(d\theta d\phi)$ of the torus and $L^p$, which map $H^p(d\theta d\phi)$ onto $H^p$. 

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(f) If $1 \leq p \leq \infty$, then

$$H^p(dm) = H^p + S^p, \quad L^p(dm) = L^p + S^p$$

where $+$ denotes algebraic direct sum and if $p=2$, each decomposition is orthogonal.

**Lemma 1.** Suppose $I^\infty = WJ^\infty$ for some inner function $W$ in $I^\infty$. Then $S^\infty$ is a weak-*closed $J^\infty$-invariant subspace of $H^\infty(m)$ such that $S^\infty = WS^\infty$.

**Proof.** By the above remark (c) and $I^\infty = WJ^\infty$, $P \ominus WP = W\mathcal{L}^2$, where $\ominus$ is orthogonal complement. Denote $S = P \ominus \sum_{f=1}^\infty W^f\mathcal{L}^2$, then $S = \bigcap_{f=1}^\infty W^fP$ and $P = S + \sum_{f=1}^\infty W^f\mathcal{L}^2$. The proof of $WS = S$ is the same as [1, p 109] and $S$ is a $J^\infty$-invariant subspace of $H^2(m)$ by that $S = \bigcap_{f=1}^\infty W^fP$ and $P$ is a $J^\infty$-invariant subspace by (2) of theorem 1. The proof of $S = S^2$ is trivial. By the definition, $S^\infty = S^2 \cap L^\infty(m)$ and hence $S^\infty = WS^\infty$ and $S^\infty$ is a $J^\infty$-invariant subspace.

**Theorem 2.** The following properties for $H^\infty(m)$ are equivalent.

1. $H^\infty(m)$ is isomorphic to $H^\infty(d\theta d\phi)$.
2. (a) $J^\infty$ has no doubly invariant subspace, (b) every nonzero weak-*closed $J^\infty$-invariant subspace $M$ of $H^\infty(m)$ has the form

$$M = \chi_E F J^\infty$$

where $\chi_E$ is a characteristic function in $J^\infty$ and $F$ is a unimodular function.

**Proof.** (1) $\Rightarrow$ (2). If $M$ is a $J^\infty$-invariant subspace, then $\overline{Z}M \subseteq M$ and so $M$ is a sesqui-invariant subspace [5]. So by [5, p 473], $M = \chi_E F J^\infty$ where $\chi_E$ is a characteristic function in $\mathcal{L}^\infty$ and $F$ is a unimodular function. But we can show easily that for any characteristic function $\chi_E$, $\chi_E \in J^\infty$ if and only if $\chi_E \in \mathcal{L}^\infty$. This proves (2). (2) $\Rightarrow$ (1). By the hypothesis of (b) in (2), we can write $I^\infty = \chi_E W J^\infty$, where $\chi_E \in J^\infty$ and $W$ is a unimodular function. If $m(E) < 1$, by the remark (c), $J^\infty$ must have some doubly invariant subspace. So we can write $I^\infty = WJ^\infty$ with $W$ in $I^\infty$. For this inner function $W$, $S^\infty$ is a weak-*closed $J^\infty$-invariant subspace of $H^\infty(m)$ and $S^\infty = WS^\infty$ by [Lemma 1]. If $S^\infty \neq \{0\}$, by the hypothesis of (b) in (2), we can write $S^\infty = \chi_E F J^\infty$ where $\chi_E \in J^\infty$ and $F$ is a unimodular function. By that $\chi_E F \in \chi_E F J^\infty$ and $\overline{WS}^\infty = S^\infty$, the function $\chi_E F \overline{W}$ is in $\chi_E F J^\infty$ and hence there exists some function $g$ in $J^\infty$ such that $\chi_E = \chi_E W g$. From $W \in I^\infty$, it follows that $\chi_E W g \in I^\infty$ and hence $\chi_E \in I^\infty$. This shows that $\chi_E = 0$ by $\chi_E \in J^\infty$ and hence $S^\infty = \{0\}$. By the remark (f), $H^\infty(m) = H^\infty$ and by the remark (e), this proves (1).
$H^\infty(d\theta d\phi)$ is not maximal as a weak-*closed subalgebra of $L^\infty(d\theta d\phi)$. So it is impossible to characterize $H^\infty(d\theta d\phi)$ by the maximality. One question that arises is: is it possible to characterize $H^\infty(d\theta d\phi)$ by subalgebras of $L^\infty(m)$ which contain it? In the next section we shall answer this.

2. Subalgebras which contain $H^\infty(m)$

Let $A$ be a weak-*Dirichlet algebra of $L^\infty(m)$. We need not always the assumption such that there exists a positive nonconstant function $v$ in $L^1(m)$ such that the measure $vdm$ is multiplicative on $A$. Muhly [6] show that $H^\infty(m)$ is a maximal weak-*closed subalgebra of $L^\infty(m)$ if and only if no nonzero function in $H^\infty(m)$ can vanish on a set of positive measure. If $V$ is a weak-*closed subalgebra which is generated by $H^\infty(m)$ and $\chi_f$ for all $f \in H^\infty(m)$, then the subalgebra $V$ contains $H^\infty(m)$ and $\chi_f \in V$ for every $f \in H^\infty(m)$. We determine forms of all subalgebras which contain $V$.

**THEOREM 3.** Let $V$ be a weak-*closed subalgebra of $L^\infty(m)$ which contains $H^\infty(m)$. The following are equivalent.

1. $\chi_f \in V$ for every $f \in V$.
2. $\chi_f \in V$ for every $f \in H^\infty(m)$.
3. Each weak-*closed subalgebra $B$ of $L^\infty(m)$ that contains $V$ has the form

$$B = \chi_E V + \chi_E^* L^\infty(m)$$

for some $\chi_E \in V$.

**Proof.** (1) $\Rightarrow$ (2) trivial.

(2) $\Rightarrow$ (3). Let $K$ be an orthogonal complement of $B$ in $L^2(m)$. We may assume $K \neq \{0\}$. Let $E$ be the support set of $K$, then $\chi_E \in V$. For since $B$ contains $H^\infty(m)$, the set $K \subseteq \overline{H}_0^2$ [7, p 226]. For each $f$ in $H^2(m)$, there exists a function $g$ in $H^\infty(m)$ such that $\chi_f = \chi_g$ [6]. So if $f$ in $K$, then $\chi_f = \chi_g \in V$ by the hypothesis of (2). If $f$ and $g$ in $K$, let $F = E_f \setminus E_g$, then $\chi_F \in V$. Since $\chi_F B \subseteq B$, we can show $\chi_F K \subseteq K$ and hence $h = g + \chi_f f$ is in $K$. So if $f, g \in K$, there exists $h \in K$ with $E_h = E_f \cup E_g$. This shows that there exists a function $f$ in $K$ such that $E_f = E$ and hence $\chi_E \in V$. Since $\chi_E K = \{0\}$ and $\chi_E \in V$, we can get $B \supseteq \chi_E V + \chi_E^* L^\infty(m)$ and $\chi_E V + \chi_E^* L^\infty(m)$ is a weak-*closed subalgebra.

We shall show $B = \chi_E V + \chi_E^* L^\infty(m)$. Suppose $B \neq \chi_E V + \chi_E^* L^\infty(m)$. Just as Muhly [6], there exists a nonconstant unimodular function $q$ and $\bar{q}$ in $B$ such that $\bar{q} \notin \chi_E V + \chi_E^* L^\infty(m)$. Then $\chi_E \bar{q} \notin \chi_E V$. Let $N$ be the weak-*
closure of polynomials of \( q, \bar{q} \) and all characteristic functions in \( V \). Then \( N \) is a commutative von Neumann algebra as an algebra of operators on \( L^2(m) \). By \( \chi_E \bar{q} \notin V \), \( V \) can not contain the whole \( \chi_E N \). There exists \( \chi_{E_0} \) in \( N \) such that \( \chi_{E_0} \cap E \neq 0 \) and for any nonzero \( \chi_f \) in \( V \)

\[
\chi_{E_0} \cap E \chi_f \neq \chi_f.
\]

For suppose there exists a nonzero \( \chi_f \) in \( V \) such that \( \chi_{H \cap E} \chi_f = \chi_f \) for any \( \chi_H \) in \( N \) such that \( \chi_{H \cap E} \neq 0 \). Then \( H \cap E \supset F \) for the nonzero \( \chi_f \) in \( V \). If \( H \cap E \neq F \), since \( \chi_H \chi_{V} \notin N \) and \( \chi_H \chi_f \neq 0 \), there exists a nonzero \( \chi_f \) in \( V \) such that \( H \cap F^c \cap E \supset F' \) arguing as above. This leads to that \( \chi_{H \cap E} \notin V \) for any \( \chi_H \) in \( N \), i.e., \( \chi_E N \subseteq V \) by that \( N \) is a commutative von Neuman algebra. This contradiction shows that there exists such a \( \chi_{E_0} \) in \( N \). By \( \chi_{E_0} \in B \), it follows that \( \chi_{E_0} K \subseteq K \). If \( \chi_{E_0} K \neq \{0\} \), we can show that there exists some nonzero \( \chi_{E_0} \) in \( V \) such that \( \chi_{E_0} \chi_{F_0} = \chi_f \). \( \chi_{E_0} K = \{0\} \). Since \( m(E_0 \cap E) \geq 0 \), this contradicts that \( E \) is the support set of \( K \). Thus \( B = \chi_E V + \chi_{E_0} L^\infty(m) \).

(3) \( \Rightarrow \) (1). Suppose \( f \) in any function in \( V \). We can assume that \( 0 < \chi_f < 1 \). Let \( D = D(f) \) be the weak*-closure of \( \{fg : g \in V\} \), then \( D \subseteq V \) and the support set of \( D \) coincides with the support set of \( f \). Let \( B = \{v \in L^\infty(m) : vD \subseteq D\} \). Then \( V \subseteq B \). From the hypothesis of (3), we can write \( B = \chi_E V + \chi_{E_0} L^\infty(m) \) for some \( \chi_E \in V \). Then we can choose \( \chi_E \) in \( V \) such that \( \chi_E \chi_E V \) has no doubly invariant subspace. If \( m(E) = 0 \), then \( B = L^\infty(m) \) which means that \( D \) is doubly-invariant and hence \( \chi_f L^\infty(m) \subseteq V \). So \( \chi_f \in V \). Suppose \( m(E) > 0 \). Since \( (1 - \chi_f) L^\infty(m) \subseteq B \) and \( \chi_{E_0} L^\infty(m) \) is the maximum doubly-invariant subspace of \( B \), we have \( E = E \) for \( E' \) or \( E = E \). If \( E' \neq E \), define \( g = \chi_{E_0} f \), then the function \( g \) is in \( V \) and \( g \neq 0 \). Arguing as above, there exist a nonzero \( \chi_f \) in \( V \) such that \( E \cap E' \supset F \). This shows that \( \chi_f \in V \).

**Corollary 1.** (Muhly [6]) The following properties for \( H^\infty(m) \) are equivalent.

1. No nonzero function in \( H^\infty(m) \) can vanish on a set of positive measure.
2. \( H^\infty(m) \) is a maximal weak*-closed subalgebra of \( L^\infty(m) \).

**Proof.** (1) \( \Rightarrow \) (2). If \( f \) is any function in \( H^\infty(m) \), then \( \chi_f \equiv 0 \) or \( \chi_f \equiv 1 \) and hence \( \chi_f \in H^\infty(m) \). Apply theorem 3 with \( V = H^\infty(m) \). (2) \( \Rightarrow \) (1). The condition (3) in theorem 3 is satisfied with \( V = H^\infty(m) \) because of the maximality of \( H^\infty(m) \). Therefore \( \chi_f \in H^\infty(m) \) for every \( f \in H^\infty(m) \). But the only real valued functions in \( H^\infty(m) \) are constants, hence \( \chi_f \equiv 0 \) or \( \chi_f \equiv 1 \). If there exists a positive nonconstant function \( v \) in \( L^1(m) \) such that
the measure \( vdm \) is multiplicative on \( A \), we can choose \( J^\infty \) as \( V \). Here \( J^\infty \) is the minimum weak-*closed subalgebra of \( L^\infty(m) \) which contains \( H^\infty(m) \) properly.

**Theorem 4.** The following properties for \( H^\infty(m) \) are equivalent.

1. \( \chi_f \in J^\infty \) for every \( f \in H^\infty(m) \).
2. Each weak-*closed subalgebra \( B \) of \( L^\infty(m) \) that contains \( H^\infty(m) \) properly has the form
   \[
   B = \chi_E J^\infty + \chi_E^c L^\infty(m)
   \]
   for some \( \chi_E \in J^\infty \).

**Proof.** If \( B \supsetneq H^\infty(m) \), then \( B \supseteq J^\infty \) since \( J^\infty \) is the minimum weak-*closed subalgebra. So by theorem 3, we can get this theorem.

In section 3 we shall show that \( \chi_f \in J^\infty \) for every \( f \in H^\infty(d\theta d\phi) \), i.e. this algebra satisfies the condition (1) of theorem 4. Moreover we shall give an example (2) such that \( H^\infty(m) \) satisfies the condition (1) of theorem 4 and it is not isomorphic to \( H^\infty(d\theta d\phi) \). Now we can get the negative answer to the question raised at the end of section 1. For \( H^\infty(m) \) in example (2) and \( H^\infty(d\theta d\phi) \) have same subalgebras in the form which contain them by theorem 4.

**Corollary 2.** Suppose \( J^\infty \neq L^\infty(m) \) and \( \chi_f \in J^\infty \) for every \( f \) in \( H^\infty(m) \). Then there is no algebra which contains \( H^\infty(m) \) and is maximal among the proper weak-*closed subalgebras of \( L^\infty(m) \).

**Proof.** Suppose \( B \) is a maximal weak-*closed subalgebra of \( L^\infty(m) \) such that \( H^\infty(m) \supseteq B \supsetneq L^\infty(m) \). By theorem 4 we can write \( B = \chi_E J^\infty + \chi_E^c L^\infty(m) \) for some \( E \) such that \( m(E) > 0 \) and \( \chi_E \in J^\infty \). Then we can choose \( \chi_E \) such that \( \chi_E J^\infty \) has no doubly invariant subspace. But \( \chi_E \in J^\infty \) if and only if \( \chi_E \in \mathcal{L}^\infty \). By the remark (d) in section 1 \( \mathcal{L}^\infty \) is isomorphic to \( L^\infty(d\theta) \) of the disc. If \( F \) in \( L^\infty(d\theta) \) corresponds to \( f \in \mathcal{L}^\infty \), then \( f(x) = F(Z(x)) \) a.e. \([5, \text{Lemma } 4] \). Hence there is a measurable set \( E' \) such that \( E' \subseteq E \) and \( m(E) \neq m(E') > 0 \) and \( \chi_{E'} \in \mathcal{L}^\infty \). If \( B' = \chi_{E'} J^\infty + \chi_{E'}^c L^\infty(m) \), then \( L^\infty(m) \supsetneq [B']_2 \supsetneq [B]_2 \) by that \( \chi_{E'} J^\infty \) has no doubly invariant subspace and hence \( B \subseteq B' \supsetneq L^\infty(m) \).

**Corollary 3.** Suppose \( \chi_f \in J^\infty \) for every \( f \) in \( H^\infty(m) \). If \( B \) is a weak-*closed subalgebra of \( L^\infty(m) \) which contains \( H^\infty(m) \) and a function \( v \) such that \( \chi_E v \in J^\infty \) for any nonzero \( \chi_E \in J^\infty \), then \( B = L^\infty(m) \).

**Proof.** By theorem 4, we can write \( B = \chi_E J^\infty + \chi_E^c L^\infty(m) \) for some \( \chi_E \in J^\infty \). Since \( B \) contains \( v \), \( \chi_E v \in \chi_E J^\infty \subseteq J^\infty \). If \( m(E) > 0 \), then \( \chi_E v \notin J^\infty \) by assumption. This implies \( m(E) = 0 \), hence \( B = L^\infty(m) \).
3. Example

(1) Let \( A \) be the weak-*Dirichlet algebra on the torus which was raised at the first of section 1. Then there exist positive nonconstant functions in \( L^{1}(d\theta d\phi) \) which are multiplicative on \( A \). \( H_{0}^{\infty}(d\theta d\phi) = zH^{\infty}(d\theta d\phi) \) and \( J^{\infty} \) is the weak-*closure of \( \bigcup_{n=0}^{\infty} z^{n}H^{\infty}(d\theta d\phi) \). Then \( \chi_{f} \in J^{\infty} \) for every \( f = f(z, w) \in H^{\infty}(d\theta d\phi) \). In fact, there exist polynomials \( p_{n}(w) \) such that for almost all points \( z_{0} \) in \( T \), as \( n \to \infty \)

\[
\int_{T} |f(z_{0}, w) - p_{n}(w)|^{2} d\phi \to 0.
\]

Then it follows that \( f(z_{0}, w) = 0 \) a.e. \( d\phi \) or \( |f(z_{0}, w)| > 0 \) a.e. \( d\phi \). Let \( E_{1} = \{z_{0} \in T : |f(z_{0}, w)| > 0 \text{ a.e. } d\phi\} \). Then the set \( E_{1} \times T \) is a support set of \( f \). For every \((n, m)\) with \( m > 0 \)

\[
\int_{E_{1}} d\theta \int_{T} z^{n}w^{m} d\phi = 0.
\]

Hence \( \chi_{f} = \chi_{E_{1} \times T} \in J^{\infty} \) by that \( L^{2}(d\theta d\phi) = J^{2} + I^{2} \) and the remark (a) in Introduction. Thus by theorem 4, each weak-*closed subalgebra \( B \) of \( L^{\infty}(d\theta d\phi) \) that contains \( H^{\infty}(d\theta d\phi) \) properly has the form \( B = \chi_{E_{1} \times T} J^{\infty} + \chi_{E_{1} \times T} L^{\infty}(d\theta d\phi) \) where \( E_{1} \) is some measurable set of \( T \) and \( F_{1} = T \setminus E_{1} \). It is known [2] that there exists a maximal uniform closed subalgebra of \( C(T^{2}) \) the set of all complex-valued continuous functions on \( T^{2} \), which contains \( A \). But by corollary 2, there is no algebra which contains \( H^{\infty}(d\theta d\phi) \) and is maximal among the proper weak-*closed subalgebras of \( L^{\infty}(d\theta d\phi) \). Moreover as \( v \) in corollary 3, we can take \( u\overline{w}^{r} \) (\( r \) is a positive real number and \( u \in L^{\infty} \) and \( |u| > 0 \)), \( \chi_{E}(E = T \times E_{2}, d\phi(E_{2}) < 1) \), etc.

(2) Let \( K \) be the Bohr compactification of the real line. Let \( A \) be the algebra of continuous, complex-valued functions on \( T \times K \) which are uniform limits of polynomials in \( z^{n}\chi_{r} \), where

\[
(n, \tau) \in \Gamma = \{(n, \tau) : \tau > 0\} \cup \{(n, 0) : n \geq 0\}
\]

and denote by \( \chi_{r} \) the characters on \( K \), where \( \tau \) in the real line. Denote by \( m \) the normalized Haar measure on \( T \times K \), then \( A \) is the weak-*Dirichlet algebra of \( L^{\infty}(m) \) [5]. There exist positive nonconstant functions in \( L^{1}(m) \) which are multiplicative on \( A \). \( H_{0}^{\infty} = zH^{\infty}(m) \) and \( J^{\infty} \) is the weak-*closure of \( \bigcup_{n=0}^{\infty} z^{n}H^{\infty}(m) \). We can show that \( \chi_{f} \in J^{\infty} \) for every \( f \in H^{\infty}(m) \) as in (1).
Let $A$ be the algebra of continuous, complex-valued functions on $K \times K$ which are uniform limits of polynomials is $\chi_{\tau_{1}} \chi_{\tau_{2}}$, where

$$(\tau_{1}, \tau_{2}) \in \Gamma = \{(\tau_{1}, \tau_{2}) : \tau_{2} > 0\} \cup \{(\tau_{1}, 0) : \tau_{1} \geq 0\}$$

and denote by $\chi_{\tau_{i}}$ the characters on $K$, where $\tau_{i}$ in the real line. Denote by $m$ the normalized Haar measure on $K \times K$, then $A$ is the weak-* Dirichlet algebra of $L^{\infty}(m)$. Then there exist no positive nonconstant functions in $L^{1}(m)$ which are multiplicative on $A$. Let $V$ be the weak-* closure of $\bigcup \overline{\chi_{\tau_{i}}}H^{\infty}(m)$, then $H^{\infty}(m) \subset V \subsetneq L^{\infty}(m)$ and $V$ is a weak-*closed subalgebra. We can show that $\chi_{f} \in V$ for every $f \in H^{\infty}(m)$ as in (1). By theorem 3, we can know the form of weak-*closed subalgebras of $L^{\infty}(m)$ which contains $V$ properly.

References


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