Grassmann geometry
on the 3-dimensional non-unimodular Lie groups

Dedicated to professor Hiroyuki Tasaki on the occasion of his 60th birthday

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Abstract. We study the Grassmann geometry of surfaces when the ambient space is a 3-dimensional non-unimodular Lie group with left invariant metric. This work together with our previous papers yield a complete classification of Grassmann geometry of orbit type in all 3-dimensional homogeneous spaces.

Key words: Grassmann geometry, non-unimodular Lie group.

1. Introduction

Let $(M, g)$ be an $m$-dimensional connected Riemannian homogeneous space and $I_0(M, g)$ the identity component of the isometry group of $(M, g)$. Fix an integer $r$ such that $1 \leq r \leq m$ and consider the Grassmann bundle $\text{Gr}^r(TM)$ over $M$ which consists of all $r$-dimensional linear subspaces of the tangent spaces of $M$. Then the Lie group $I_0(M, g)$ acts on $\text{Gr}^r(TM)$ through the differentials of isometries and each $I_0(M, g)$-orbit $O$ in $\text{Gr}^r(TM)$ gives a homogeneous bundle over $M$. An $r$-dimensional connected submanifold $S$ of $M$ is called an $O$-submanifold if all tangent spaces of $S$ belong to $O$, and the collection of $O$-submanifolds is called an $O$-geometry. Such an $O$-geometry is collectively called the Grassmann geometry of orbit type. A typical example of $O$-submanifold is an (extrinsic) homogeneous submanifold which is defined as an orbit $G(p)$ in $M$ by a subgroup $G$ of $I_0(M, g)$ where $p \in M$.

One of the fundamental problem in the Grassmann geometry of orbit type is to determine, whether a given $O$-geometry is empty or not.

When we found nonempty $O$-geometry, then the following problem naturally arises:

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“Are there somewhat canonical submanifolds (e.g., totally geodesic submanifolds, minimal submanifolds or submanifolds of specific curvature etc) in the prescribed \( \mathcal{O}\)-geometry?”

From these viewpoints, Grassmann geometry of Riemannian symmetric space has been studied by H. Naitoh [20] and J. Berndt-J. H. Eschenburg-H. Naitoh-K. Tsukada [1].

On the other hand, Grassmann geometry of orbit type in 3-dimensional Riemannian homogeneous manifolds has been initiated by K. Kuwabawa and the present authors. More precisely, Grassmann geometry on the Heisenberg group \( H_3 \), the motion groups \( E(2) \) and \( E(1, 1) \) were investigated in [8], [15], respectively. These three Riemannian homogeneous spaces are realized as unimodular Lie groups with left invariant metric.

In our previous papers [10], [11], we investigated Grassmann geometry of orbit type on the other unimodular Lie groups \( SU(2), SL(2, \mathbb{R}) \).

As is well known, simply connected and connected 3-dimensional homogeneous Riemannian manifolds are Riemannian symmetric spaces or simply connected Lie groups with left invariant metric [24].

In addition, we have studied all the non-empty \( \mathcal{O}\)-geometries of surfaces on the 3-dimensional unimodular Lie groups in our previous works [8], [10], [11], [15] and it is known in [21] that all the \( \mathcal{O}\)-geometries of hypersurfaces on Riemannian symmetric spaces with an arbitrary dimension are non-empty. Hence the only still remaining 3-dimensional Riemannian homogeneous spaces are 3-dimensional non-unimodular Lie groups.

The purpose of this paper is to study Grassmann geometry of orbit type in 3-dimensional non-unimodular Lie groups.

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2. Left invariant metrics on non-unimodular Lie groups

In the following we recall some fundamental properties about non-unimodular Lie groups discussed in J. Milnor [19].

2.1.

Let \( G \) be a non-unimodular 3-dimensional Lie group with a left invariant metric \( g \) and with its Lie algebra \( \mathfrak{g} \). Moreover let \( \langle \cdot, \cdot \rangle \) be the inner product on \( \mathfrak{g} \) induced from \( g \). Then the unimodular kernel \( \mathfrak{u} \) of \( \mathfrak{g} \) is defined by
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\[ u = \{ X \in g \mid \text{tr} \, \text{ad}(X) = 0 \}. \]

Here \( \text{ad} : g \to \text{End}(g) \) is the \textit{adjoint representation} of \( g \), that is, a homomorphism defined by

\[ \text{ad}(X)Y = [X,Y]. \]

The non-unimodular property of \( G \) implies that \( u \) is a 2-dimensional ideal of \( g \) which contains the ideal \( [g,g] \).

On \( g \), we can take an orthonormal basis \( \{E_1, E_2, E_3\} \) such that

1. \( \langle E_1, X \rangle = 0 \) for all \( X \in u \),
2. \( \langle [E_1, E_2], [E_1, E_3] \rangle = 0 \),

where \( \{E_2, E_3\} \) is an orthonormal basis of \( u \).

The representation matrix \( A \) of \( \text{ad}(E_1) : u \to u \) relative to the basis \( \{E_2, E_3\} \) is expressed as

\[ A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}. \]

Then the commutation relations of the basis are given by

\[ [E_1, E_2] = \alpha E_2 + \beta E_3, \quad [E_2, E_3] = 0, \quad [E_1, E_3] = \gamma E_2 + \delta E_3, \]

with \( \text{tr} \, A = \alpha + \delta \neq 0 \) and \( \alpha \gamma + \beta \delta = 0 \). These commutation relations imply that \( g \) is solvable.

Under a suitable homothetic change of the metric, we may assume that \( \alpha + \delta = 2 \). Then the constants \( \alpha, \beta, \gamma \) and \( \delta \) are represented as

\[ \alpha = 1 + \xi, \quad \beta = (1 + \xi)\eta, \quad \gamma = -(1 - \xi)\eta, \quad \delta = 1 - \xi. \]

If necessarily, by changing the sign of \( E_1, E_2 \) and \( E_3 \), we may assume that the constants \( \xi \) and \( \eta \) satisfy the condition \( \xi, \eta \geq 0 \). We note that for the case that \( \xi = \eta = 0 \), \( (G,g) \) has constant negative curvature (see Example 2.6).

For more informations on left invariant metrics on 3-dimensional non-unimodular Lie groups, we refer to [4], [14], [23].

From now on we work under this normalization. Then the commutation
relations are rewritten as
\[ [E_1, E_2] = (1 + \xi)\{E_2 + \eta E_3\}, \quad [E_2, E_3] = 0, \quad [E_3, E_1] = (1 - \xi)\{\eta E_2 - E_3\}. \]

We refer \((\xi, \eta)\) as the structure constants of the non-unimodular Lie algebra \(g\).

Non-unimodular Lie algebras \(g = g(\xi, \eta)\) are classified by the Milnor invariant \(D := \det A = (1 - \xi^2)(1 + \eta^2)\). More precisely we have the following fact (cf \([18], [19]\)).

**Proposition 2.1** For any pair of \((\xi, \eta)\) and \((\xi', \eta')\) which are not \((0, 0)\), two Lie algebras \(g(\xi, \eta)\) and \(g(\xi', \eta')\) are isomorphic if and only if their Milnor invariants \(D\) and \(D'\) agree.

**Remark 2.2** In \([25]\), H. Tasaki and M. Umehara introduced an invariant of 3-dimensional Lie algebras equipped with inner products. Their invariant \(\chi(g)\) for the non-unimodular Lie algebra \(g = g(\xi, \eta)\) is \(4/D\). Note that in case \(D = 0\), \(\chi(g)\) is regarded as \(\infty\).

The characteristic equation for the matrix \(A\) is \(\lambda^2 - 2\lambda + D = 0\). Thus the eigenvalues of \(A\) are \(1 \pm \sqrt{1 - D}\). Roughly speaking, this means that the moduli space of non-unimodular Lie algebras \(g(\xi, \eta)\) may be decomposed into three kinds of types which are determined by the conditions \(D > 1\), \(D = 1\) and \(D < 1\).

The Levi-Civita connection of \((G, g)\) is given by the following table:

**Proposition 2.3**
\[
\begin{align*}
\nabla_{E_1} E_1 &= 0, & \nabla_{E_1} E_2 &= \eta E_3, & \nabla_{E_1} E_3 &= -\eta E_2 \\
\nabla_{E_2} E_1 &= -(1 + \xi)E_2 - \xi \eta E_3, & \nabla_{E_2} E_2 &= (1 + \xi)E_1, & \nabla_{E_2} E_3 &= \xi \eta E_1 \\
\nabla_{E_3} E_1 &= -\eta E_2 - (1 - \xi)E_3, & \nabla_{E_3} E_2 &= \xi \eta E_1, & \nabla_{E_3} E_3 &= (1 - \xi)E_1.
\end{align*}
\]

The Riemannian curvature \(R\) is given by
\[
\begin{align*}
R(E_1, E_2)E_1 &= \{\xi \eta^2 + (1 + \xi)^2 + \xi \eta^2(1 + \xi)\}E_2, \\
R(E_1, E_2)E_2 &= -\{\xi \eta^2 + (1 + \xi)^2 + \xi \eta^2(1 + \xi)\}E_1, \\
R(E_1, E_3)E_1 &= -\{\xi \eta^2 - (1 - \xi)^2 + \xi \eta^2(1 - \xi)\}E_3.
\end{align*}
\]
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$$R(E_1, E_3)E_3 = \{\xi\eta^2 - (1 - \xi)^2 + \xi\eta^2(1 - \xi)\}E_1,$$
$$R(E_2, E_3)E_2 = \{1 - \xi^2(1 + \eta^2)\}E_3,$$
$$R(E_2, E_3)E_3 = -\{1 - \xi^2(1 + \eta^2)\}E_2$$

and $R(E_i, E_j)E_k = 0$ if otherwise.

The basis $\{E_1, E_2, E_3\}$ diagonalizes the Ricci quadratic form $r$. The principal Ricci curvatures are given by

$$r(E_1, E_1) = -2\{1 + \xi^2(1 + \eta^2)\} < -2,$$
$$r(E_2, E_2) = -2\{1 + \xi(1 + \eta^2)\} < -2,$$
$$r(E_3, E_3) = -2\{1 - \xi(1 + \eta^2)\}.$$ (2.1)

The scalar curvature $\rho$ is

$$\rho = -2\{3 + \xi^2(1 + \eta^2)\} < 0.$$

**Remark 2.4** The sign of $r(E_3, E_3)$ can be positive, null or negative. In fact, if $D < 0$, then $r(E_3, E_3) > 0$. If $D \geq 0$, then $r(E_3, E_3) = 0$ is possible. In case $D > 0$, there exists a metric of strictly negative curvature. Moreover if $D > 1$, there exists a metric of constant negative curvature.

Let us orient $g$ so that $\{E_1, E_2, E_3\}$ is positive. Then the cross product $\times$ with respect to this orientation defines a linear operator $L$ on $g$ by

$$L(u \times v) = [u, v], \quad u, v \in g.$$

Then we obtain

$$L(E_1) = 0,$$
$$L(E_2) = (1 - \xi)\eta E_2 + (\xi - 1)E_3,$$
$$L(E_3) = (1 + \xi)E_2 + (1 + \xi)\eta E_3.$$

Since $g$ is non-unimodular, $L$ is not self-adjoint.

**Proposition 2.5** Let $(G, g)$ be a 3-dimensional non-unimodular Lie group and use the notations introduced above. Then $G$ is locally symmetric if and only if $\xi = 0$ or $(\xi, \eta) = (1, 0)$. 
Proof. Direct computation of $\nabla R$ yields the result. For instance, we have

$$
(\nabla_{E_2} R)(E_2, E_3)E_1 = -(\nabla_{E_2} R)(E_3, E_2)E_1
= 2\xi^2\eta(1 + \xi)(1 + \eta^2)E_2 + \xi(1 + \xi)(1 - \xi)(1 + \eta^2)E_3.
$$

Other cases of $\nabla_{E_2} R$ are zero. From these one can see that $\nabla_{E_i} R = 0$ if and only if $\xi = 0$ or $\eta = 0$. It is a straightforward manner to check the equivalence $\nabla_{E_i} R = 0$ ($i = 1, 3$) and $\xi = 0$ or $(\xi, \eta) = (1, 0)$. □

2.2.

Here we give explicit matrix group models of non-unimodular Lie groups.

The simply connected Lie group $\tilde{G}$ corresponding to the non-unimodular Lie algebra $\mathfrak{g}$ with structure constants $(\xi, \eta)$ is given explicitly by

$$
\tilde{G}(\xi, \eta) = \left\{ \begin{pmatrix}
1 & 0 & 0 & x \\
0 & \alpha_{11}(x) & \alpha_{12}(x) & y \\
0 & \alpha_{21}(x) & \alpha_{22}(x) & z \\
0 & 0 & 0 & 1
\end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\},
$$

where $\alpha_{ij}(x)$ is the $(i, j)$-entry of $\exp(xA)$. This shows that $\tilde{G}(\xi, \eta)$ is the semi-direct product $\mathbb{R} \ltimes \mathbb{R}^2$ with multiplication

$$
(x, y, z) \cdot (x', y', z') = (x + x', y + \alpha_{11}(x)y' + \alpha_{12}(x)z', z + \alpha_{21}(x)y' + \alpha_{22}(x)z'). \quad (2.2)
$$

The Lie algebra of $\tilde{G}(\xi, \eta)$ is spanned by the basis

$$
E_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 1 + \xi & -(1 - \xi)\eta & 0 \\
0 & (1 + \xi)\eta & 1 - \xi & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
E_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad E_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}.
$$
This basis satisfies the commutation relations

\[
[E_1, E_2] = (1 + \xi)\{E_2 + \eta E_3\}, \quad [E_2, E_3] = 0, \quad [E_3, E_1] = (1 - \xi)\{\eta E_2 - E_3\}.
\]

Thus the Lie algebra of \(\tilde{G}(\xi, \eta)\) is the non-unimodular Lie algebra \(g = g(\xi, \eta)\) with structure constants \((\xi, \eta)\). The left invariant vector fields corresponding to \(E_1, E_2\) and \(E_3\) are

\[
E_1 = \frac{\partial}{\partial x}, \quad E_2 = \alpha_{11}(x) \frac{\partial}{\partial y} + \alpha_{21}(x) \frac{\partial}{\partial z}, \quad E_3 = \alpha_{12}(x) \frac{\partial}{\partial y} + \alpha_{22}(x) \frac{\partial}{\partial z}.
\]

**Example 2.6** (\(\xi = 0, \mathcal{D} \geq 1\)) The simply connected Lie group \(\tilde{G}(0, \eta)\) is isometric to the hyperbolic 3-space \(H^3(-1)\) and given explicitly by

\[
\tilde{G}(0, \eta) = \left\{ \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & e^x \cos(\eta x) & -e^x \sin(\eta x) & y \\ 0 & e^x \sin(\eta x) & e^x \cos(\eta x) & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| x, y, z \in \mathbb{R} \right\}.
\]

The left invariant metric is \(dx^2 + e^{-2x}(dy^2 + dz^2)\). Thus \(\tilde{G}(0, \eta)\) is the warped product model of \(H^3(-1)\). In fact, put \(w = e^x\) then the left invariant metric of \(\tilde{G}(0, \eta)\) is rewritten as the Poincaré metric

\[
\frac{dy^2 + dz^2 + dw^2}{w^2}.
\]

The Milnor invariant of \(\tilde{G}(0, \eta)\) is \(\mathcal{D} = 1 + \eta^2 \geq 1\).

**Example 2.7** (\(\eta = 0, \mathcal{D} \leq 1\)) For each \(\xi \geq 0\), \(\tilde{G}(\xi, 0)\) is given by:

\[
\tilde{G}(\xi, 0) = \left\{ \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & e^{(1+\xi)x} & 0 & y \\ 0 & 0 & e^{(1-\xi)x} & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| x, y, z \in \mathbb{R} \right\}.
\]

The left invariant Riemannian metric is given explicitly by

\[
dx^2 + e^{-2(1+\xi)x} dy^2 + e^{-2(1-\xi)x} dz^2.
\]
The Milnor invariant is $\mathcal{D} = 1 - \xi^2 \leq 1$.

This family of Riemannian homogenous spaces has been studied in [6], [7], [9]. Here we observe locally symmetric examples:

- If $\xi = 0$ then $\tilde{G}(0,0)$ is a warped product model of hyperbolic 3-space $H^3(-1)$.
- If $\xi = 1$ then $\tilde{G}(1,0)$ is isometric to the Riemannian product $H^2(-4) \times \mathbb{R}$ of hyperbolic plane $H^2(-4)$ of curvature $-4$ and the real line. In fact, via the coordinate change $(u,v) = (2y, e^{2x})$, the metric is rewritten as

$$\frac{du^2 + dv^2}{4v^2} + dz^2.$$

**Example 2.8** ($\xi = 1$, $\mathcal{D} = 0$) Assume that $\xi = 1$. Then $\tilde{G}(1,\eta)$ is given explicitly by

$$\tilde{G}(1,\eta) = \left\{ \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & e^{2x} & 0 & y \\ 0 & \eta(e^{2x} - 1) & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| \begin{array}{c} x, y, z \in \mathbb{R} \end{array} \right\}.$$

The left invariant metric is

$$dx^2 + \{ e^{-4x} + \eta^2(1 - e^{-2x})^2 \} dy^2 - 2\eta(1 - e^{-2x}) dy dz + dz^2.$$

The non-unimodular Lie group $\tilde{G}(1,\eta)$ has sectional curvatures

$$K_{12} = -3\eta^2 - 4, \quad K_{13} = K_{23} = \eta^2,$$

where $K_{ij}$ ($i \neq j$) denote the sectional curvatures of the planes spanned by vectors $E_i$ and $E_j$. One can check that $\tilde{G}(1,\eta)$ is isometric to the so-called *Bianchi-Cartan-Vranceanu space* $M^3(-4,\eta)$ with 4-dimensional isometry group and isotropy subgroup $SO(2)$:

$$M^3(-4,\eta) = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 < 1, ds^2 \},$$

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2} + \left( dz + \frac{\eta(ydx - xdy)}{1 - x^2 - y^2} \right)^2$$
with \( \eta \geq 0 \). The family \( \{ \tilde{G}(1, \eta) \}_{\eta \geq 0} \) is characterized by the condition \( D = 0 \). In particular \( M^3(-4, \eta) \) with positive \( \eta \) is isometric to the universal covering \( \tilde{SL}(2, \mathbb{R}) \) of the special linear group equipped with the above metric (cf. [26]), but not isomorphic to \( \tilde{SL}(2, \mathbb{R}) \) as Lie groups. We here note that \( SL(2, \mathbb{R}) \) is a unimodular Lie group, while \( \tilde{G}(1, \eta) \) is non-unimodular.

We now explain the difference of the case \( D \geq 1 \) and \( D < 1 \) defined by the Milnor invariant \( D \) from the viewpoint of Ricci curvature, more precisely, existence of solvsolitons. In the last part of this paper we shall also see the difference of these cases from the viewpoint of Grassmann geometry.

It is known that any nontrivial homogeneous Ricci soliton must be non-compact, expanding and non-gradient. Particular examples of homogeneous Ricci soliton are solvsolitons. Let \( G \) be a simply connected solvable Lie group equipped with a left invariant metric. Then the left invariant metric is said to be a solvsoliton [5], [16] (also called solsoliton) if its Ricci operator at the unit element \( e \) has the form

\[
\text{Ric} = c \text{Id} + \partial,
\]

where \( c \) is a constant and \( \partial \) is a derivation of the Lie algebra \( g = T_e G \). On the simply connected non-unimodular Lie group \( G \) with structure constants \( (\xi, \eta) \), the existence of solvsolitons are described as follows (see [5], [16]):

- The case that \( D \geq 1 \): The only solvsolitons are of constant negative curvature.
- The case that \( D < 1 \): There exist non-constant curvature solvsolitons.

Let us describe solvsolitons on \( G \) more explicitly. The solvsoliton equation \( \text{Ric} = c \text{Id} + \partial \) has solutions if and only if

- The case that \( \xi = 0, c = -2, \partial = 0 \). In this case, \( G \) is the hyperbolic 3-space \( H^3(-1) \) and satisfies \( D \geq 1 \) (See Example 2.6).
- The case that \( \xi = 1, c = -2(\eta^2 + 2) \) and \( \partial \) is defined by

\[
\partial E_1 = \partial E_2 = 0, \quad \partial E_3 = 4(\eta^2 + 1)E_3.
\]

In this case \( G \) has 4-dimensional isometry group and \( D = 0 \). The principal Ricci curvatures have the signature \((-,-,+)\) (See Example 2.8).
• The case that $\eta = 0$, $c = -2(1 + \xi^2)$ and $\partial$ is defined by

$$
\partial E_1 = 0, \quad \partial E_2 = 2\xi(\xi - 1)E_2, \quad \partial E_3 = 2\xi(\xi + 1)E_3.
$$

In this case $D = 1 - \xi^2 \leq 1$. In particular, the metric is of constant curvature when $\xi = 0$ (See Example 2.7). Note that if $0 \leq \xi < 1$, $G$ has strictly negative curvature.

**Corollary 2.9** Let $G = \tilde{G}(\xi, \eta)$ be a simply connected 3-dimensional non-unimodular Lie group with structure constants $(\xi, \eta)$ with $\xi \notin \{0, 1\}$. Then $\tilde{G}(\xi, \eta)$ is solvsoliton when and only when $\eta = 0$, i.e., $D \leq 1$.

### 3. Grassmann geometry

#### 3.1.

Let us consider the classification of non-empty $O$-geometries on all the 3-dimensional Riemannian homogeneous spaces. As described in Introduction, a 3-dimensional Riemannian homogeneous space is locally isometric to either a Riemannian symmetric space or a Lie group with left invariant metric. Moreover, the non-empty $O$-geometries on the 3-dimensional unimodular groups with left invariant metrics have been determined, and it is known that all the $O$-geometries of hypersurfaces on Riemannian symmetric spaces are non-empty (cf. Remark 3.1). Hence, summing up Proposition 2.5, Examples 2.6, Examples 2.7 and 2.8, we may restrict our attention to non-unimodular Lie groups with $\xi \notin \{0, 1\}$.

**Remark 3.1** In the paper [21] the following has been shown: Let $M$ be a compact simply connected Riemannian symmetric space. Then, any $O$-geometry of hypersurfaces on $M$ is non-empty. However, this assumption on $M$ is not essential and thus we can easily extend the result into the general case where $M$ is just a locally Riemannian symmetric space.

**Proposition 3.2** Every non-unimodular Lie group $G$ with structure constants $(\xi, \eta)$ with $\xi \notin \{0, 1\}$ has trivial isotropy.

**Proof.** According to a classification of Riemannian 3-manifolds with 4-dimensional isometry group, due to L. Bianchi [2] and E. Cartan [3], The only possibility for a non-unimodular Lie group $G$ has non-trivial isotropy is either $\xi = 0$ or $\xi = 1$. Hence under the condition $\xi \notin \{0, 1\}$, $G$ has trivial isotropy. □
3.2.

Now we study orbit-type Grassmann geometry of surfaces in the non-unimodular Lie group \((G, g)\) with structure constants \((\xi, \eta)\) so that \(\xi \notin \{0, 1\}\).

Denote by \(I_o(G, g)\) the identity component of the isometry group of \((G, g)\). We take an orbit \(O\) in \(\text{Gr}^2(TG)\) under the action of \(I_o(G, g)\). As described in Introduction, \(\text{Gr}^2(TG)\) is the Grassmann bundle over \(G\) of all 2-planes tangent to \(G\) and \(I_o(G, g)\) naturally acts on \(\text{Gr}^2(TG)\) through isometries of \((G, g)\). Since \((G, g)\) is a Riemannian homogeneous space, the orbit \(O\) is a homogeneous bundle over \(G\) with respect to \(I_o(G, g)\). In this case an \(O\)-surface is by definition a connected surface \(S\) in \(G\) such that \(T_xS \in O\) for any \(x \in S\) and the \(O\)-geometry is the collection of such the \(O\)-surfaces.

Under the condition \(\xi \notin \{0, 1\}\), \(I_o(G, g)\) is 3-dimensional. Thus \(I_o(G, g)\) consists of all left translations of \(G\). Hence an \(I_o(G, g)\)-orbit \(O\) induces a distribution on \(G\) which is constructed by the left translations of the unique plane \(\Pi \subset T_eG\) which belongs to \(O\). Thus the orbit space of \(I_o(G, g)\)-action is diffeomorphic to the Grassmann manifold \(\text{Gr}^2(T_eG)\) of all 2-planes in \(T_e(G)\), which is also diffeomorphic to the real projective plane \(\mathbb{RP}^2\). Hence an \(O\)-geometry is not empty if and only if the distribution on \(G\) corresponding to \(O\) is involutive.

Here note that since \((G, g)\) is Riemannian homogeneous, the existence of \(O\)-surfaces is equivalent to the local existence of involutive distributions all of whose leaves are \(O\)-surfaces.

3.3.

Before starting the classification of \(O\)-surfaces, here we give some examples.

Example 3.3 (The canonical normal subgroup) Let \(U\) be the (maximal) integral surface of the distribution spanned by \(E_2\) and \(E_3\). Then \(U\) is a flat surface with constant mean curvature 1. Note that this distribution is obtained by left translation of the unimodular kernel \(u\), so \(U\) is a normal subgroup of \(G\). In particular, when \(\xi = 0\), \(U\) is locally isometric to a horosphere of a hyperbolic 3-space \(H^3(-1)\). In the universal covering \(\tilde{G}(\xi, \eta)\), \(U\) is realized as
Note that when $\mathcal{D} > 1$, $G$ has no 2-dimensional Lie subgroups other than $U$ ([18, Theorem 3.6]).

**Example 3.4** When $\eta = 0$, the distributions spanned by $\{E_1, E_2\}$ and $\{E_1, E_3\}$ are involutive. The (maximal) integral surfaces of these distributions are totally geodesic and of constant curvature $-(1 + \xi)^2$, $-(1 - \xi)^2$, respectively.

**Theorem 3.5** ([12]) Let $G$ be a 3-dimensional non-unimodular Lie group with left invariant metric $g$ and structure constants $(\xi, \eta)$. Assume that $\xi \notin \{0,1\}$. If $S$ is a surface with parallel second fundamental form in $G$, then there are two possibilities:

- $S$ is an integral surface of the distribution spanned by $\{E_2, E_3\}$.
- $S$ is an integral surface of the distributions spanned by $\{E_1, E_2\}$, respectively $\{E_1, E_3\}$.

The latter case only occurs if $\eta = 0$.

Among surfaces in $G$ with $\xi \notin \{0,1\}$, surfaces with parallel second fundamental form are also characterized as constant mean curvature surfaces with vertically harmonic Gauss map ([13]).

**Corollary 3.6** ([12]) Let $G$ be a 3-dimensional non-unimodular Lie group with left invariant metric $g$ and structure constants $(\xi, \eta)$ and $\xi \notin \{0,1\}$. Then $G$ admits totally geodesic surfaces when and only when $\eta = 0$. The only totally geodesic surfaces are integral surfaces of the distributions spanned by $\{E_1, E_2\}$ or $\{E_1, E_3\}$.

The class $\eta = 0$ is characterized by the existence of totally geodesic surfaces. In the case of 3-dimensional non-unimodular Lie group $G$, the umbilicity of surfaces does not always imply the parallelism of the second fundamental form. Recently J. Manzano and R. Souam classified totally umbilic surfaces in $G$.

**Theorem 3.7** ([17]) Let $G$ be a 3-dimensional non-unimodular Lie group
with left invariant metric $g$ and structure constants $(\xi, \eta)$ and $\xi \not\in \{0, 1\}$. Then $G$ admits totally umbilic surfaces when and only when $\eta = 0$. Totally umbilic surfaces are locally isometric to

- a totally geodesic surface or
- a surface invariant by a 1-parameter group of isometries associated to one of the Killing vector fields $E_2$ or $E_3$ which has non-constant mean curvature.

**Remark 3.8** Let $(M, g)$ be a submanifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$. Then $M$ is said to be an extrinsic sphere if it is totally umbilic and has parallel mean curvature vector field. One can see that extrinsic spheres have parallel second fundamental form. A totally umbilic submanifold has parallel second fundamental form if and only if it is an extrinsic sphere. The only extrinsic spheres in the non-unimodular Lie group $G$ with $\xi \not\in \{0, 1\}$ are totally geodesic surfaces.

### 3.4.

Now let $S^2(g)$ be the unit sphere in the Lie algebra $g = g(\xi, \eta)$ centered at the origin. For a unit vector $W \in S^2(g)$, one can associate a linear plane $P(W) \subset g$ orthogonal to $W$. Then the mapping $P : S^2(g) \to \text{Gr}^2(g)$ induces the bijection $P : \mathbb{R}P^2(g) \to \text{Gr}^2(g)$. Here we introduce a curvature function $\kappa : S^2(g) \to \mathbb{R}$ by

$$\kappa(W) = \text{sectional curvature of } P(W).$$

Express $W$ as $W = w_1 E_1 + w_2 E_2 + w_3 E_3$. Then the curvature function $\kappa(W)$ is computed as

$$\kappa(W) = -\{\xi^2(1 + \eta^2) + 3\} + 2\{1 + \xi^2(1 + \eta^2)\}w_1^2 + \{1 + \xi(1 + \eta^2)\}w_2^2 + \{1 - \xi(1 + \eta^2)\}w_3^2.$$  

$$\text{(3.3)}$$

Now let us denote by $O(P(W))$ the $I_0(G, g)$-orbit containing the 2-plane $P(W)$ for $W \in S^2(g)$. We have

$$L(W) = \{-w_2(\xi - 1)\eta + w_3(1 + \xi)\}E_2 + \{w_2(\xi - 1) + w_3(1 + \xi)\eta\}E_3.$$ 

Thus we have
\[ \langle L(W), W \rangle = \eta(w_2^2 + w_3^2) + \xi\{2w_2w_3 + \eta(-w_2^2 + w_3^2)\} . \]

The condition \( \langle L(W), W \rangle = 0 \) holds if and only if the distribution on \( G \), defined by the left translations of the plane \( P(W) \) in \( \mathfrak{g} \), is involutive. We have the following criterion.

**Proposition 3.9**  Assume that \( \xi \notin \{0, 1\} \). Then there exists an \( \mathcal{O}(P(W)) \)-surface if and only if

\[ \eta(w_2^2 + w_3^2) + \xi\{2w_2w_3 + \eta(-w_2^2 + w_3^2)\} = 0, \quad w_1^2 + w_2^2 + w_3^2 = 1. \quad (3.4) \]

**Example 3.10**  Take \( W = \pm E_1 \). Then \( W \) satisfies (3.4) for any \( \xi > 0 \) and \( \eta \geq 0 \). As we have seen in Example 3.3, \( \mathcal{O}(P(W)) \)-surfaces are flat and of constant mean curvature 1. Moreover they are parallel.

**Example 3.11**  Take \( W = \pm E_2 \) (resp. \( \pm E_3 \)). Then \( W \) satisfies (3.4) for \( \xi \notin \{0, 1\} \) if and only if \( \eta = 0 \). In case \( \eta = 0 \), as we have seen before in Example 3.4, the \( \mathcal{O}(P(W)) \)-surfaces are totally geodesic and of constant curvature \(-(1 - \xi)^2 < 0 \) (resp. \(-(1 + \xi)^2 < 0 \)).

3.5.

Now we classify \( \mathcal{O} \)-surfaces. Define a matrix \( B \) by

\[ B = \begin{pmatrix} \eta(1 - \xi) & \xi \\ \xi & \eta(1 + \xi) \end{pmatrix} . \]

Then the criterion equation (3.4) can be rewritten as

\[ (w_2, w_3)B \begin{pmatrix} w_2 \\ w_3 \end{pmatrix} = 0 \quad (3.5) \]

and \( w_1^2 = 1 - w_2^2 - w_3^2 \). The eigenvalues of \( B \) are

\[ \lambda_+ = \eta + \xi \sqrt{1 + \eta^2} > 0, \quad \lambda_- = \eta - \xi \sqrt{1 + \eta^2} . \]

Here we note that

\[ D - 1 = (\eta + \xi \sqrt{1 + \eta^2}) (\eta - \xi \sqrt{1 + \eta^2}) = \lambda_+ \lambda_- . \]
Thus, under the assumption $\xi \not\in \{0, 1\}$, $\lambda_+ > 0$ and hence the signature of $\mathcal{D} - 1$ coincides with that of $\lambda_-$. The $\lambda_+$-eigenvector has the form

$$w_3 = (\eta + \sqrt{1 + \eta^2}) w_2.$$ 

The $\lambda_-$-eigenvector has the form

$$w_3 = (\eta - \sqrt{1 + \eta^2}) w_2.$$ 

To obtain solutions to (3.5), we choose a rotation of axes in $g(\xi, \eta)$ which diagonalizes the matrix $B$ as follows:

$$w_1 = v_1$$

$$w_2 = \cos \phi v_2 - \sin \phi v_3$$

$$w_3 = \sin \phi v_2 + \cos \phi v_3,$$

where $v_1^2 + v_2^2 + v_3^2 = 1$. Here $\phi$ is a constant angle determined by the condition $\cot(2\phi) = -\eta < 0$ and $\pi/4 < \phi < \pi/2$. Note that when $\eta = 0$, we regard $\phi = \pi/4$. Under this choice, we have

$$\cos(2\phi) = -\frac{\eta}{\sqrt{1 + \eta^2}}, \quad \sin(2\phi) = \frac{1}{\sqrt{1 + \eta^2}}.$$ 

With respect to the new coordinates $(v_2, v_3)$, the equation (3.5) is rewritten as

$$\lambda_+ v_2^2 + \lambda_- v_3^2 = 0.$$ 

### 3.6. $\mathcal{D} > 1$

First we consider the case $\mathcal{D} > 1$. In this case both the eigenvalues of $B$ is positive, that is, $B$ is a positive definite symmetric matrix. Then the only solution $(w_2, w_3)$ for (3.5) is $w_2 = w_3 = 0$. This means that $W = \pm E_1$. Hence $O$-surfaces have parallel second fundamental form (see Example 3.10).

**Proposition 3.12** In the non-unimodular Lie group $G$ satisfying $\xi \not\in \{0, 1\}$ and $\mathcal{D} > 1$, $O(P(W))$-surfaces exist if and only if $W = \pm E_1$. 

3.7. $\mathcal{D} = 1$

Next we consider the case $\mathcal{D} = 1$, i.e., $\lambda_- = 0$. Since $\xi > 0$, we have $0 < \xi = \eta / \sqrt{1 + \eta^2} < 1$ and $\lambda_+ = 2\eta > 0$.

Since we are working with the condition $\mathcal{D} = 1$, (3.5) is reduced to $v_2 = 0$. Hence

$$w_2 = -\sin \phi v_3, \quad w_3 = \cos \phi v_3.$$ 

Namely

$$w_3 = -\cot \phi w_2.$$ 

Here we notice that

$$\tan \phi = \eta + \sqrt{1 + \eta^2}.$$ 

Hence

$$w_2 = -(\eta + \sqrt{1 + \eta^2}) w_3.$$ 

Equivalently,

$$w_3 = (\eta - \sqrt{1 + \eta^2}) w_2.$$ 

Namely $(w_2, w_3)$ is a $\lambda_-$-eigenvector of $B$.

**Proposition 3.13** In the non-unimodular Lie group $G$ satisfying $\xi \notin \{0, 1\}$ and $\mathcal{D} = 1$, $\mathcal{O}(P(W))$-surfaces exist if and only if $W$ satisfies

$$w_3 = (\eta - \sqrt{1 + \eta^2}) w_2, \quad \eta > 0, \quad w_1^2 + w_2^2 + w_3^2 = 1,$$

where the case $w_2 = w_3 = 0$ implies that $W = \pm E_1$.

3.8. $\mathcal{D} < 1$

Finally we consider the case $\xi \notin \{0, 1\}$ and $\mathcal{D} < 1$. We exclude the case $\mathcal{D} = 0$ since in this case $\xi = 1$. Note that this case contains the case $\eta = 0$.

In the case, $\mathcal{D} < 1$ and $\mathcal{D} \neq 0$, we have $\lambda_- = \eta - \xi \sqrt{1 + \eta^2} < 0$. Hence it suffices to solve

$$\lambda_+ v_2^2 - (-\lambda_-) v_3^2 = 0.$$
This is equivalent to
\[ \sqrt{\lambda_+} v_2 \pm \sqrt{-\lambda_-} v_3 = 0. \]

Namely
\[ \sqrt{\lambda_+} (\cos \phi w_2 + \sin \phi w_3) \pm \sqrt{-\lambda_-} (-\sin \phi w_2 + \cos \phi w_3) = 0. \]

In case \( \eta = 0 \), this equation is reduced to \( w_2 = 0 \) or \( w_3 = 0 \). The case \( \eta = 0 \) will be investigated in Example 3.16. Here we assume that \( \eta > 0 \).

Under this assumption, we get
\[ \frac{w_3}{w_2} = -\frac{\sqrt{\lambda_+} \mp \tan \phi \sqrt{-\lambda_-}}{\sqrt{\lambda_+} \tan \phi \pm \sqrt{-\lambda_-}} \]
\[ = -\frac{\sqrt{\xi \sqrt{1 + \eta^2 + \eta} \mp (\eta + \sqrt{1 + \eta^2}) \sqrt{\xi \sqrt{1 + \eta^2} - \eta}}{\sqrt{\xi \sqrt{1 + \eta^2 + \eta}(\eta + \sqrt{1 + \eta^2}) \pm \sqrt{\xi \sqrt{1 + \eta^2} - \eta}} w_2, \]

Thus the vector \( W = w_1 E_1 + w_2 E_2 + w_3 E_3 \) with
\[ w_3 = -\frac{\sqrt{\xi \sqrt{1 + \eta^2 + \eta} \mp (\eta + \sqrt{1 + \eta^2}) \sqrt{\xi \sqrt{1 + \eta^2} - \eta}}{\sqrt{\xi \sqrt{1 + \eta^2 + \eta}(\eta + \sqrt{1 + \eta^2}) \pm \sqrt{\xi \sqrt{1 + \eta^2} - \eta}} w_2, \]
\[ w_1^2 + w_2^2 + w_3^2 = 1 \]
satisfies (3.5).

**Proposition 3.14** In the non-unimodular Lie group \( G \) satisfying \( \xi \notin \{0, 1\} \), \( \eta > 0 \) and \( D < 1 \), \( \mathcal{O}(P(W)) \)-surfaces exist if and only if \( W \) satisfies
\[ w_3 = -\frac{\sqrt{\xi \sqrt{1 + \eta^2 + \eta} \mp (\eta + \sqrt{1 + \eta^2}) \sqrt{\xi \sqrt{1 + \eta^2} - \eta}}{\sqrt{\xi \sqrt{1 + \eta^2 + \eta}(\eta + \sqrt{1 + \eta^2}) \pm \sqrt{\xi \sqrt{1 + \eta^2} - \eta}} w_2, \]
\[ w_1^2 + w_2^2 + w_3^2 = 1 \]
or \( W = \pm E_1 \).
Example 3.15 ($\eta = \xi > 0$) To confirm the classification of $O$-surfaces with $D < 1$, $D \neq 0$ and $\eta > 0$, here we consider the case $\eta = \xi > 0$. In this case we have $\lambda_- < 0$ and hence $D < 1$. The solutions to (3.5) are

$$w_2 = w_3 = 0, \quad w_3 = -w_2 \neq 0 \quad \text{or} \quad w_3 = \frac{\eta - 1}{\eta + 1}w_2.$$ 

Hence there exist three kinds of $O$-surfaces, the normal subgroup $U$, $O$-surfaces determined by $w_3 = -w_2$ or $w_3 = ((\eta - 1)/(\eta + 1))w_2$.

Example 3.16 ($\eta = 0$) Let us observe the case $\xi \notin \{0, 1\}$ and $\eta = 0$. Then $D < 1$ and the criterion equation (3.4) is reduced to $w_2w_3 = 0$. Hence $W$ can be represented as

$$W = \cos \theta \, E_1 + \sin \theta \, E_3 \quad \text{or} \quad W = \cos \theta \, E_1 + \sin \theta \, E_2$$

for some constant $\theta$. Thus $W$ makes constant angle with $E_1$. Hence $O$-surfaces are examples of the so-called constant angle surfaces in $G$. A. Nistor showed that constant angle surfaces in $G$ with $\eta = 0$ are expressed as products of two appropriate curves [22, Theorem 4.1]. We investigate these $O$-surfaces in detail.

In case $W = \cos \theta \, E_1 + \sin \theta \, E_3$, we can take orthonormal frame field $\{X_1, X_2, X_3 = W\}$ along the $O$-surface as

$$X_1 = -\sin \theta \, E_1 + \cos \theta \, E_3, \quad X_2 = E_2.$$ 

Direct computations show that

$$\nabla_{X_1}X_3 = -\cos \theta(1 - \xi)X_1, \quad \nabla_{X_2}X_3 = -\cos \theta(1 + \xi)X_2.$$ 

Thus the shape operator derived from $X_3$ is

$$\begin{pmatrix} \cos \theta(1 - \xi) & 0 \\ 0 & \cos \theta(1 + \xi) \end{pmatrix}.$$ 

Thus the mean curvature and the Gauss curvature are given by

$$H = \cos \theta, \quad K = -\sin^2 \theta(1 + \xi)^2.$$ 

Moreover in the semi-direct product model $\mathbb{R} \ltimes \mathbb{R}^2 = \tilde{G}(\xi, 0)$, the $O$-surface
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is given explicitly as the product of two curves \( \gamma_1(v) \cdot \gamma_2(u) \) with

\[
\gamma_1(v) = (0, v, 0), \\
\gamma_2(u) = \left(-\sin \theta u, 0, -\frac{\cot \theta}{1 - \xi} \exp\{- \left(1 - \xi \right) \sin \theta u\}\right)
\]

for \( 0 < \theta < \pi \) and \( \theta \neq \pi/2 \).

If \( \theta = 0 \), then \( W = E_1 \) and hence the \( O \)-surface is the canonical normal subgroup \( U \) with Lie algebra \( u \). On the other hand, if \( \theta = \pi/2 \), then \( W = E_3 \) and \( O \)-surfaces are integral surfaces of \( \text{span}\{E_1, E_2\} \). These \( O \)-surfaces are totally geodesic and of constant curvature \( K = -(1 + \xi^2) \) (see Example 3.4).

Summing up the classification procedure above, we obtain the main result of this paper.

**Theorem 3.17** Let \( G \) be a 3-dimensional non-unimodular Lie group with left invariant metric \( g \) and structure constants \( (\xi, \eta) \) such that \( \xi \notin \{0, 1\} \). Then

- If \( D > 1 \), then the only \( O \)-surfaces are the canonical normal subgroup \( U \) in \( G \) with the unimodular kernel \( u \) as its Lie subalgebra and the surfaces obtained by translating \( U \) by the left action of \( G \) where \( O = O(P(W)) \) with \( W = \pm E_1 \).
- If \( D = 1 \), then the orbits \( O \) for which there exist \( O \)-surfaces are the orbits \( O(P(W)) \) determined by

\[
w_3 = \left(\eta - \sqrt{1 + \eta^2}\right)w_2, \quad \eta > 0, \quad w_1^2 + w_2^2 + w_3^2 = 1.
\]

These orbits construct one circle through \( O(P(\pm E_1)) \) in the orbit space \( \mathbb{R}P^2(g) \).
- If \( D < 1 \), then the orbits \( O \) for which there exist \( O \)-surfaces are the orbits \( O(P(W)) \) determined by

\[
w_3 = -\frac{\sqrt{x^2 + \eta^2} + \eta \pm (\eta + \sqrt{1 + \eta^2})\sqrt{x^2 + \eta^2} - \eta}{\sqrt{x^2 + \eta^2} + \eta(\eta + \sqrt{1 + \eta^2}) \pm \sqrt{x^2 + \eta^2} - \eta} w_2,
\]

\[
w_1^2 + w_2^2 + w_3^2 = 1
\]
\(-\) when \(\eta = 0\),

\[
W = \cos \theta E_1 + \sin \theta E_3, \text{ or } W = \cos \theta E_1 + \sin \theta E_2
\]

for some constant angle \(\theta\) such that \(0 < \theta < \pi\).

These orbits construct two circles through \(\mathcal{O}(P(\pm E_1))\) in the orbit space \(\mathbb{R}P^2(\mathfrak{g})\).

References


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