The critical values of $L$-functions of base change for Hilbert modular forms

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(Received January 14, 2017; Revised June 5, 2017)

Abstract. In this paper we generalize some results, obtained by Shimura, Yoshida and the author, on critical values of $L$-functions of $l$-adic representations attached to Hilbert modular forms twisted by finite order characters, to the critical values of $L$-functions of arbitrary base change to totally real number fields of $l$-adic representations attached to Hilbert modular forms twisted by some general finite-dimensional representations.

Key words: $L$-functions, special values, Hilbert modular forms.

1. Introduction

For $F$ a totally real number field of degree $n$, let $J_F$ be the set of infinite places of $F$, and let $\Gamma_F := \text{Gal}(\mathbb{Q}/F)$. Let $f$ be a normalized Hecke eigenform of $\text{GL}(2)/F$ of weight $k = (k(\tau))_{\tau \in J_F}$, where all $k(\tau)$ have the same parity and $k(\tau) \geq 2$. We denote by $\Pi$ the cuspidal automorphic representation of $\text{GL}(2)/F$ generated by $f$. In this paper we assume that $\Pi$ is non-CM. We denote by $\rho_\Pi$ the $l$-adic representation of $\Gamma_F$ attached to $\Pi$. Define $k_0 = \max\{k(\tau) | \tau \in J_F\}$ and $k^0 = \min\{k(\tau) | \tau \in J_F\}$. Any integer $m \in \mathbb{Z}$ such that $(k_0 - k^0)/2 < m < (k_0 + k^0)/2$ is called a critical value for $f$ or $\Pi$.

Let $F'$ be a totally real finite extension of $F$. Consider a finite-dimensional continuous representation

$$\psi : \Gamma_{F'} \to \text{GL}_N(\mathbb{C}).$$

Let $V_\psi$ be the space corresponding to $\psi$. We denote by $d^+_{\tau'}(\psi)$ the dimension of the subspace of $V_\psi$ on which the complex conjugation corresponding to $\tau' \in J_{F'}$ acts by $+1$, and by $d^-_{\tau'}(\psi)$ the dimension of the subspace of $V_\psi$ on which the complex conjugation corresponding to $\tau' \in J_{F'}$ acts by $-1$. Throughout this paper we write $a \sim b$ for $a, b \in \mathbb{C}$ if $b \neq 0$ and $a/b \in \mathbb{Q}$.

2010 Mathematics Subject Classification : 11F41, 11F80, 11R42, 11R80.
In this article we prove the following result:

**Theorem 1.1** Assume \( k(\tau) \geq 3 \) for all \( \tau \in J_F \) and \( k(\tau) \mod 2 \) is independent of \( \tau \). Let \( F' \) be a totally real finite extension of \( F \). Let \( \psi \) be a finite-dimensional complex-valued continuous representation of \( \Gamma_{F'} \) such that \( K := \overline{\mathbb{Q}}^{ker\psi} \) is an abelian extension of a totally real number field. Then

\[
L(m, \rho|_{\Gamma_{F'}} \otimes \psi) \sim \pi^{m[F':\mathbb{Q}]} \dim \psi \prod_{\tau' \in J_{F'}} c_{\tau'|F}^{(-1)^{(m+1)}} (\Pi)^{d_{\tau'}(\psi)} (\Pi)^{d_{\tau'}(\psi)}
\]

for any integer \( m \) satisfying

\[
(k_0 + 1)/2 \leq m < (k_0 + k^0)/2,
\]

where \( c_{\tau'|F}^{-} (\Pi) \) and \( c_{\tau'|F}^{+} (\Pi) \) appear in Propositions 2.2 and 2.3 below.

Theorem 1.1 is a generalization of Theorem 4.3 of [S], the main theorem of [Y], Theorem 4 of [Y], Theorems 1, 1.2 and 1.3 of [V1] (i.e. Propositions 2.1, 2.2 and 2.3 below; when \( \psi \) is abelian, Theorem 1.1 could be deduced easily from Propositions 2.1, 2.2 and 2.3 below), and of [V4]. It is conjectured that the result obtained in Theorem 1.1 should be true for arbitrary finite-dimensional complex-valued continuous representations \( \psi \) of \( \Gamma_{F'} \).

We remark that Theorem 1.1 above was proved by Shimura and Yoshida (see Theorem 4.3 of [S], and the Main theorem of [Y]) when \( \psi \) has dimension 1, for any integer \( m \) satisfying \((k_0 - k^0)/2 < m < (k_0 + k^0)/2\) (i.e. for all the critical values \( m \)), but because in this paper we use Brauer’s induction theorem, we have to restrict ourselves to the integers \( m \) satisfying \((k_0 + 1)/2 < m < (k_0 + k^0)/2\), in order to make sure that our \( L \)-functions that have negative exponent do not vanish at \( m \) (see Section 3 below for details).

2. **Known results**

Consider \( F \) a totally real number field and let \( J_F \) be the set of infinite places of \( F \). If \( \Pi \) is a cuspidal automorphic representation (discrete series at infinity) of weight \( k = (k(\tau))_{\tau \in J_F} \) of \( GL(2)/F \), where all \( k(\tau) \) have the same parity and all \( k(\tau) \geq 2 \), then there exists ([T]) a \( \lambda \)-adic representation

\[
\rho_{\Pi} := \rho_{\Pi,\lambda} : \Gamma_F \to GL_2(O_{\lambda}) \hookrightarrow GL_2(\overline{\mathbb{Q}}_l),
\]
which satisfies \( L(s, \rho_{\Pi, \lambda}) = L(s - (k_0 - 1)/2, \Pi) = L(s - (k_0 - 1)/2, f) \) (the equality up to finitely many Euler factors) and is unramified outside the primes dividing \( \mathfrak{n}l \) (by fixing a specific isomorphism \( i : \mathbb{Q}_l \sim \mathbb{C} \) one can regard \( \rho_{\Pi} \) as a complex-valued representation). Here \( O \) is the coefficients ring of \( \Pi \) and \( \lambda \) is a prime ideal of \( O \) above some prime number \( l \), \( \mathfrak{n} \) is the level of \( \Pi \) and \( f \) is the normalized Hecke eigenform of \( \text{GL}(2)/F \) of weight \( k \) corresponding to \( \Pi \). We denote by \( F^\times_\infty \) the archimedean part of the idele group \( F^\times_\mathfrak{a} \) of \( F \).

We know (this is Theorem 1.1 of [V1], which is a generalization of Theorem 4.3 of [S]):

**Proposition 2.1** Assume \( k(\tau) \geq 3 \) for all \( \tau \in J_F \) and \( k(\tau) \text{ mod } 2 \) is independent of \( \tau \). Let \( F' \) be a totally real finite extension of \( F \). Then for every \( \epsilon = (\epsilon(\tau)) \in (\mathbb{Z}/2\mathbb{Z})^{J_{F'}} \), there exists a constant \( u(\epsilon, \Pi) \in \mathbb{C}^\times/\mathbb{Q}^\times \) with the following property. If \( \psi \) is a finite order Hecke character of \( F' \) such that

\[
\psi_\infty(x) = \prod_{\tau \in J_{F'}} \text{sgn}(x_\tau)^{\epsilon(\tau)+m}, \quad x = (x_\tau) \in F^\times_\infty,
\]

then

\[
L(m, \rho_{\Pi}|_{\Gamma_{F'}, \otimes \psi}) \sim \pi^{m[F':\mathbb{Q}]} u(\epsilon, \Pi)
\]

for any integer \( m \) satisfying

\[
(k_0 + 1)/2 \leq m < (k_0 + k^0)/2.
\]

We know (this is Theorem 1.2 of [V1], which is a generalization of the main theorem of [Y]):

**Proposition 2.2** Assume that \( k(\tau) \geq 3 \) for all \( \tau \in J_F \) and \( k(\tau) \text{ mod } 2 \) is independent of \( \tau \). Let \( F' \) be a totally real finite extension of \( F \). Then, for every \( \tau \in J_{F'} \), there exist constants \( c^\tau_\mathfrak{a}(\Pi) \in \mathbb{C}^\times \) which are determined uniquely \( \text{mod } \mathbb{Q}^\times \) such that

\[
u(\epsilon, \Pi) \sim \prod_{\tau \in J_{F'}} c^{\epsilon(\tau)}_\mathfrak{a}(\Pi),
\]

where \( \epsilon = (\epsilon(\tau)) \in (\mathbb{Z}/2\mathbb{Z})^{J_{F'}} \), and \( u(\epsilon, \Pi) \) was defined in Proposition 2.1.
Here we understand that $c_0^0(\Pi) = c_+^+(\Pi)$, $c_1^1(\Pi) = c_-^-(\Pi)$ by identifying $\mathbb{Z}/2\mathbb{Z}$ with $\{0, 1\}$.

We know (this is Theorem 1.3 of [V1], which is a generalization of Theorem 4 of [Y]):

**Proposition 2.3** Assume that $k(\tau) \geq 3$ for all $\tau \in J_F$ and $k(\tau) \mod 2$ is independent of $\tau$. Let $F'$ be a totally real finite extension of $F$. Then we have

$$c_\tau^\pm(\Pi) = c_{\tau|F'}^\pm(\Pi) \text{ for every } \tau \in J_F'.$$

3. The proof of Theorem 1.1

We fix a non-CM cuspidal automorphic representation $\Pi$ of $\text{GL}(2)/F$ as in Theorem 1.1, and let $F'/F$ be a totally real finite extension. Let $\psi$ be a finite-dimensional representation of $\Gamma_{F'}$ as in Theorem 1.1, such that $K := \mathbb{Q}^{\ker \psi}$ is an abelian extension of a totally real number field. We denote by $F''$ the maximal totally real subfield of $K := \mathbb{Q}^{\ker \psi}$. Obviously $F''/F'$ is Galois and $K$ is an abelian extension of $F''$.

From the beginning of §15 of [CR] we know that there exist some subfields $F_i \subseteq F''$ such that $\text{Gal}(F''/F_i)$ are cyclic, and some integers $n_i$, such that the trivial representation

$$1_{F'} : \text{Gal}(F''/F') \to \mathbb{C}^\times$$

can be written as

$$[F'': F']1_{F'} = \sum_{i=1}^{u} n_i \text{Ind}_{\text{Gal}(F''/F_i)}^{\text{Gal}(F''/F')} 1_{F_i},$$

where $1_{F_i} : \text{Gal}(F''/F_i) \to \mathbb{C}^\times$ is the trivial representation. In particular we have $[F'' : F'] = \sum_{i=1}^{u} n_i [F_i : F']$. Then

$$L(s, \rho_{\Pi}|_{\Gamma_{F'}}, \otimes \psi)[F'':F'] = \prod_{i=1}^{u} L(s, \rho_{\Pi}|_{\Gamma_{F'}}, \otimes \psi \otimes \text{Ind}_{\Gamma_{F_i}}^{\Gamma_{F'}} 1_{F_i})^{n_i}$$

$$= \prod_{i=1}^{u} L(s, \text{Ind}_{\Gamma_{F_i}}^{\Gamma_{F'}} (\rho_{\Pi}|_{\Gamma_{F_i}} \otimes \psi|_{\Gamma_{F_i}}))^{n_i}$$
critical values of \( L \)-functions

\[
\prod_{i=1}^{u} L(s, \rho_{\Pi}|_{\Gamma_{F_i}} \otimes \psi|_{\Gamma_{F_i}})^{n_i} = \prod_{i=1}^{u} L(s, \rho_{\Pi}|_{\Gamma_{F_i}} \otimes \psi|_{\Gamma_{F_i}})^{n_i}.
\]

We write

\[
\psi|_{\Gamma_{F_i}} = \bigoplus_{j=1}^{u_i} \psi_{ij},
\]

where \( \psi_{ij} \) are irreducible representations of \( \Gamma_{F_i} \). Since \( \text{Gal}(F''/F_i) \) is cyclic, \( \psi_{ij}|_{\Gamma_{F''}} \) is abelian and \( \psi_{ij} \) is irreducible, we get that \( \psi_{ij} \simeq \text{Ind}_{\Gamma_{F_i}}^{\Gamma_{F''}} \phi_{ij} \) for some continuous character

\[
\phi_{ij} : \Gamma_{F_i} \to \mathbb{C}^*,
\]

where \( F_{ij} \) is a subfield of \( F'' \) which contains \( F_i \) (This is true, because if \( \sigma \) be a generator of \( \text{Gal}(F''/F_i) \), then since \( F''/F_i \) is Galois, \( \sigma \) permutes the irreducible components of \( \psi_{ij}|_{\Gamma_{F''}} \). The representation \( \psi_{ij}|_{\Gamma_{F''}} \) is abelian, and thus a direct sum of characters. Let \( \phi \) be one of these characters. We denote by \( F_{ij} \) the subfield of \( F'' \) which contains \( F_i \) having the property that \( \text{Gal}(F''/F_{ij}) \) is the stabiliser of \( \phi \) under the action of \( \text{Gal}(F''/F_i) = \langle \sigma \rangle \). The character \( \phi \) extends to a character \( \phi_{ij} \) of \( \Gamma_{F_{ij}} \). Then, because \( \psi_{ij} \) is irreducible, \( \sigma \in \text{Gal}(F_{ij}/F_i) \) permutes simply-transitively all the components of the representation \( \psi_{ij}|_{\Gamma_{F_{ij}}} \), so that it is a sum of conjugates of \( \phi_{ij} \) and hence abelian, and we have that \( [F_{ij} : F_i] = \dim \psi_{ij} \). Let \( V_{\psi_{ij}} \) be the space corresponding to \( \psi_{ij} \), and \( V_{\phi_{ij}} \) be the space corresponding to \( \phi_{ij} \).

Since \( \text{Hom}_{\Gamma_{F_{ij}}} (V_{\psi_{ij}}, V_{\phi_{ij}}) \) is nontrivial, by Frobenius reciprocity we get that \( \text{Hom}_{\Gamma_{F_i}} (V_{\psi_{ij}}, \text{Ind}_{\Gamma_{F_{ij}}}^{\Gamma_{F_i}} V_{\phi_{ij}}) \) is also non-trivial. But \( \dim \text{Ind}_{\Gamma_{F_{ij}}}^{\Gamma_{F_i}} \phi_{ij} = \dim \psi_{ij} \), and thus we obtain \( \psi_{ij} \simeq \text{Ind}_{\Gamma_{F_{ij}}}^{\Gamma_{F_i}} \phi_{ij} \). Therefore we have

\[
L(s, \rho_{\Pi}|_{\Gamma_{F''}} \otimes \psi)^{[F''/F_i]} = \prod_{i=1}^{u} L(s, \rho_{\Pi}|_{\Gamma_{F_i}} \otimes \psi|_{\Gamma_{F_i}})^{n_i} = \prod_{i=1}^{u} \prod_{j=1}^{u_i} L(s, \rho_{\Pi}|_{\Gamma_{F_{ij}} \otimes \phi_{ij}})^{n_i}.
\]
Using the potential modularity of the representation \( \rho_\Pi|_{\Gamma_{F_{ij}}} \) (see Theorem A of [BGGT], Theorem 2.1 of [V2] or Theorem 1.1 of [V3]), one can prove easily the meromorphic continuation to the entire complex plane of the functions \( L(s, \rho_\Pi|_{\Gamma_{F_{ij}}} \otimes \phi_{ij}) \) (for details see for example the proof of Theorem 1.1 of [V1]), and hence one gets the meromorphic continuation to the entire complex plane of the function \( L(s, \rho_\Pi|_{\Gamma_{F'}} \otimes \psi)^{[F'' : F']} \). Moreover, from the proof of Theorem 1.1 of [V1] we know that the function \( L(s, \rho_\Pi|_{\Gamma_{F_{ij}}} \otimes \phi_{ij}) \) has no poles or zeros at \( s = m \) for each integer \( m \) satisfying \((k_0 + 1)/2 \leq m < (k_0 + k^0)/2\). Thus for each integer \( m \) satisfying \((k_0 + 1)/2 \leq m < (k_0 + k^0)/2\), we get the identity

\[
L(m, \rho_\Pi|_{\Gamma_{F'}} \otimes \psi)^{[F'' : F']} = \prod_{i=1}^{u_i} \prod_{j=1}^{u_j} L(m, \rho_\Pi|_{\Gamma_{F_{ij}}} \otimes \phi_{ij})^{n_i}.
\]

We have:

\[
[F'' : F'] \psi = \sum_{i=1}^{u_i} n_i \text{Ind}_{\text{Gal}(K/F')} \phi_{iF_i} = \sum_{i=1}^{u_i} n_i \text{Ind}_{\text{Gal}(K/F_i)} \left( \sum_{j=1}^{u_j} \psi_{ij} \right)
\]

\[
= \sum_{i=1}^{u_i} n_i \text{Ind}_{\text{Gal}(K/F_i)} \left( \sum_{j=1}^{u_j} \text{Ind}_{\Gamma_{F_{ij}}} \phi_{ij} \right)
\]

\[
= \sum_{i=1}^{u_i} \sum_{j=1}^{u_j} n_i \text{Ind}_{\text{Gal}(K/F_{ij})} \phi_{ij}.
\]

Thus

\[
[F'' : F'] d_-^-(\psi) = \sum_{i=1}^{u_i} \sum_{j=1}^{u_j} n_i \sum_{\tau_{ij} \in J_{F_{ij}} | \tau_{ij} |_{F'} = \tau'} d_-(\phi_{ij})
\]

and

\[
[F'' : F'] d_+^-(\psi) = \sum_{i=1}^{u_i} \sum_{j=1}^{u_j} n_i \sum_{\tau_{ij} \in J_{F_{ij}} | \tau_{ij} |_{F'} = \tau'} d_+^-(\phi_{ij}),
\]

for any \( \tau' \in J_{F'} \). Also we have that \([F'' : F'] \text{dim} \psi = \sum_{i=1}^{u_i} \sum_{j=1}^{u_j} n_i [F_{ij} : \psi]\).
critical values of L-functions

$F'$, and thus $[F'' : \mathbb{Q}] \dim \psi = \sum_{i=1}^{u} \sum_{j=1}^{u_i} n_i [F_{ij} : \mathbb{Q}]$.

Now from Propositions 2.1, 2.2 and 2.3 one gets easily that

$$L(m, \rho_\Pi|_{\Gamma_{F_{ij}}} \otimes \phi_{ij})$$

$$\sim \pi^m [F_{ij} : \mathbb{Q}] \prod_{\tau_{ij} \in J_{F_{ij}}} c_{\tau_{ij}}(\phi_{ij})^{(m+1)} \left(\Pi\right) d_{\tau_{ij}}(\phi_{ij}) c_{\tau_{ij}}(\phi_{ij})^{m} d_{\tau_{ij}}(\phi_{ij})$$

and hence

$$L(m, \rho_\Pi|_{\Gamma_{F'}} \otimes \psi)[F'' : F']$$

$$= \prod_{i=1}^{u} \prod_{j=1}^{u_i} L(m, \rho_\Pi|_{\Gamma_{F_{ij}}} \otimes \phi_{ij})^{n_i}$$

$$\sim \prod_{i=1}^{u} \prod_{j=1}^{u_i} \pi^m [F_{ij} : \mathbb{Q}] \prod_{\tau_{ij} \in J_{F_{ij}}} c_{\tau_{ij}}(\phi_{ij})^{(m+1)} \left(\Pi\right) d_{\tau_{ij}}(\phi_{ij}) c_{\tau_{ij}}(\phi_{ij})^{m} d_{\tau_{ij}}(\phi_{ij})$$

$$= \pi^m [F'' : \mathbb{Q}] \dim \psi \prod_{\tau' \in J_{F'}} c_{\tau'}(\phi_{ij})^{(m+1)} \left(\Pi\right) \sum_{i=1}^{u} \sum_{j=1}^{u_i} n_i \sum_{\tau_{ij} \in J_{F_{ij}}} c_{\tau_{ij}}(\phi_{ij})^{m} d_{\tau_{ij}}(\phi_{ij})$$

$$\times \prod_{\tau' \in J_{F'}} c_{\tau'}(\phi_{ij})^{m} \left(\Pi\right) \sum_{i=1}^{u} \sum_{j=1}^{u_i} n_i \sum_{\tau_{ij} \in J_{F_{ij}}} c_{\tau_{ij}}(\phi_{ij})^{m} d_{\tau_{ij}}(\phi_{ij})$$

$$= \pi^m [F'' : \mathbb{Q}] \dim \psi \prod_{\tau' \in J_{F'}} c_{\tau'}(\phi_{ij})^{(m+1)} \left(\Pi\right) \left[F' : F''\right] d_{\tau'}(\psi) c_{\tau'}(\phi_{ij})^{m} \left(\Pi\right) \left[F'' : F'\right] d_{\tau'}(\psi)$$

$$= \pi^m [F'' : \mathbb{Q}] \dim \psi \prod_{\tau' \in J_{F'}} c_{\tau'}(\phi_{ij})^{(m+1)} \left(\Pi\right) \left[F' : F''\right] d_{\tau'}(\psi) c_{\tau'}(\phi_{ij})^{m} \left(\Pi\right) \left[F'' : F'\right] d_{\tau'}(\psi)$$

and thus

$$L(m, \rho_\Pi|_{\Gamma_{F'}} \otimes \psi) \sim \pi^m [F'' : \mathbb{Q}] \dim \psi \prod_{\tau' \in J_{F'}} c_{\tau'}(\phi_{ij})^{(m+1)} \left(\Pi\right) d_{\tau'}(\psi) c_{\tau'}(\phi_{ij})^{m} \left(\Pi\right) d_{\tau'}(\psi),$$

which proves Theorem 1.1. □
References


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