On the annihilators of formal local cohomology modules

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Abstract. Let $a$ denote an ideal in a commutative Noetherian local ring $(R, \mathfrak{m})$ and $M$ a non-zero finitely generated $R$-module of dimension $d$. Let $d := \dim(M/aM)$. In this paper we calculate the annihilator of the top formal local cohomology module $\mathfrak{H}^d_a(M)$. In fact, we prove that $\text{Ann}_R(\mathfrak{H}^d_a(M)) = \text{Ann}_R(M/U_R(a, M))$, where

$$U_R(a, M) := \cup\{N : N \subseteq M \text{ and } \dim(N/aN) < \dim(M/aM)\}.$$ 

We give a description of $U_R(a, M)$ and we will show that

$$\text{Ann}_R(\mathfrak{H}^d_a(M)) = \text{Ann}_R(M/G_R(a, M)).$$

where $G_R(a, M)$ denotes the largest submodule of $M$ such that $\text{Ass}_R(M) \cap V(a) \subseteq \text{Ass}_R(M/G_R(a, M))$ and $\text{Ass}_R(M)$ denotes the set $\{p \in \text{Ass}_M : \dim R/p = \dim M\}$.

Key words: attached primes, local cohomology, annihilator.

1. Introduction

Throughout this paper, $R$ is a commutative Noetherian ring with identity, $a$ is an ideal of $R$ and $M$ is a non-zero finitely generated $R$-module. Recall that the $i$-th local cohomology module of $M$ with respect to $a$ is defined as

$$H^i_a(M) := \varprojlim_{n \geq 1} \text{Ext}_R^i(R/a^n, M).$$

For basic facts about commutative algebra see [7] and [11]; for local cohomology refer to [6].

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Let $a$ be an ideal of a commutative Noetherian local ring $(R, \mathfrak{m})$ and $M$ a non-zero finitely generated $R$-module. For each $i \geq 0$; $\mathfrak{F}^i_a(M) := \lim_{\rightarrow} H^i_{\mathfrak{m}}(M/a^nM)$ is called the $i$-th formal local cohomology of $M$ with respect to $a$. The basic properties of formal local cohomology modules are found in [1], [5], [9], [12] and [14].

In [14] Schenzel investigated the structure of formal local cohomology modules and gave the upper and lower vanishing and non-vanishing to these modules. In particular, he proved that $\text{Sup}\{i \in \mathbb{Z} : \mathfrak{F}^i_a(M) \neq 0\} = \dim(M/\mathfrak{a}M)$. Thus $\mathfrak{F}^{\dim(M)}_a(M) \neq 0$ if and only if $\dim(M/\mathfrak{a}M) = \dim M$ (cf. [14, 4.5]).

For an $R$-module $M$ and an ideal $a$, the cohomological dimension of $M$ with respect to $a$ is defined as $\text{cd}(a, M) := \max\{i \in \mathbb{Z} : H^i_a(M) \neq 0\}$. For more details see [8]. For any ideal $a$ of $R$, the radical of $a$, denoted by $\sqrt{a}$, is defined to be the set $\{x \in R : x^n \in a$ for some $n \in \mathbb{N}\}$.

A non-zero $R$-module $M$ is called secondary if its multiplication map by any element $a$ of $R$ is either surjective or nilpotent. A secondary representation for an $R$-module $M$ is an expression for $M$ as a finite sum of secondary modules. If such a representation exists, we will say that $M$ is representable. A prime ideal $p$ of $R$ is said to be an attached prime of $M$ if $p = (N :_R M)$ for some submodule $N$ of $M$. If $M$ admits a reduced secondary representation, $M = S_1 + S_2 + \ldots + S_n$, then the set of attached primes $\text{Att}_R(M)$ of $M$ is equal to $\{\sqrt{0 :_R S_i} : i = 1, \ldots, n\}$ (see [10]).

Recall that $\text{Ass}_R(M)$ denotes the set $\{p \in \text{Ass} M : \dim R/p = \dim M\}$. It is well known that $\text{Att}_R \mathfrak{F}^{\dim M}_a(M) = \{p \in \text{Ass}_R(M) : p \supseteq a\}$ (cf. [5, Theorem 3.1]).

There are many results about annihilators of local cohomology modules. For example see [2], [3] and [4]. The following theorem is a main result of [2] about the annihilators of the top local cohomology modules.

**Theorem 1.1** ([2, Theorem 2.3]) Let $R$ be a Noetherian ring and $a$ an ideal of $R$. Let $M$ be a non-zero finitely generated $R$-module such that $\text{cd}(a, M) = \dim M$. Then $\text{Ann}_R H^i_a(M) = \text{Ann}_R(M/T_R(a, M))$, where

$$T_R(a, M) := \bigcup\{N : N \leq M \text{ and } \text{cd}(a, N) < \text{cd}(a, M)\}.$$

Note that, for a local ring $(R, \mathfrak{m})$, we have $\text{cd}(\mathfrak{m}, M) = \dim M$ (cf. [8]). Thus
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\[ T_R(m, M) := \cup \{N : N \leq M \text{ and } \dim N < \dim M\}, \]

which is the largest submodule of \( M \) such that \( \dim(T_R(m, M)) < \dim(M) \).

Here, by using the above main result, we obtain some results about annihilators of top formal local cohomology modules. In Section 2, at first we define a new notation \( U_R(a, M) \) and we prove the following Theorem which is a main result of this paper.

**Theorem 1.2** Let \( a \) be an ideal of a local ring \((R, m)\) and \( M \) a finitely generated \( R \)-module of dimension \( d \) such that \( \mathfrak{F}_a^d(M) \neq 0 \). Then

\[ \text{Ann}_R \mathfrak{F}_a^d(M) = \text{Ann}_R M / U_R(a, M), \]

where \( U_R(a, M) := \cup \{N : N \leq M \text{ and } \dim(N/aN) < \dim(M/aM)\} \).

In Section 3, we obtain the radical of the annihilator of top formal local cohomology module \( \mathfrak{F}_a^{\dim M}(M) \). For this we define notation \( G_R(a, M) \) and we obtain the following main result.

**Theorem 1.3** Let \( a \) be an ideal of a local ring \((R, m)\) and \( M \) a finitely generated \( R \)-module of dimension \( d \) such that \( \mathfrak{F}_a^d(M) \neq 0 \). Then

\[ \sqrt{\text{Ann}_R \mathfrak{F}_a^d(M)} = \text{Ann}_R M / G_R(a, M), \]

where \( G_R(a, M) \) denotes the largest submodule of \( M \) such that \( \text{Assh}_R(M) \cap V(a) \subseteq \text{Ass}_R(M / G_R(a, M)) \).

### 2. Annihilators of the top formal local cohomology modules

Let \( a \) be an ideal of a local ring \((R, m)\) and \( M \) a finitely generated \( R \)-module of dimension \( d \) such that \( \dim(M/aM) = d \). In this section, we will calculate the annihilator of the formal local cohomology module \( \mathfrak{F}_a^d(M) \). Note that the assumption \( \dim(M/aM) = d \) implies that \( \mathfrak{F}_a^d(M) \neq 0 \) by (cf. [14, 4.5]).

**Definition 2.1** Let \( a \) be an ideal of \( R \) and \( M \) be a non-zero finitely generated \( R \)-module. We denote by \( U_R(a, M) \) the largest submodule of \( M \) such that \( \dim(U_R(a, M)/aU_R(a, M)) < \dim(M/aM) \). One can check that

\[ U_R(a, M) := \cup \{N : N \leq M \text{ and } \dim(N/aN) < \dim(M/aM)\}. \]
The following lemma is needed in this section.

**Lemma 2.2** Let \((R, \mathfrak{m})\) be a local ring and \(\mathfrak{a}\) an ideal of \(R\). Let \(M\) be a finitely generated \(R\)-module of finite dimension \(d\) such that \(\dim(M/\mathfrak{a}M) = d\). Then

i) \(M/U_R(\mathfrak{a}, M)\) has no non-zero submodule of dimension less than \(d\);

ii) \(\text{Ass}_R(M/U_R(\mathfrak{a}, M)) = \text{Att}_R \mathfrak{F}_\mathfrak{a}^d(M)\);

iii) \(\text{Ass}_R U_R(\mathfrak{a}, M) = \text{Ass}_R M - \text{Att}_R \mathfrak{F}_\mathfrak{a}^d(M)\);

iv) \(\mathfrak{F}_\mathfrak{a}^d(M) \cong \mathfrak{F}_\mathfrak{a}^d(M/U_R(\mathfrak{a}, M)) \cong \mathbb{H}^d_{\mathfrak{m}}(M/U(\mathfrak{a}, M))\).

**Proof.** Let \(U := U_R(\mathfrak{a}, M)\).

i) Suppose that \(L\) is a submodule of \(M\) such that \(U \subseteq L \subseteq M\) and \(\dim(L/U) < d\). We will show that \(U = L\). By [14, Theorem 1.1] and [14, Theorem 3.11], the short exact sequence

\[
0 \to U \to L \to L/U \to 0
\]

induces an exact sequence

\[
\cdots \to \mathfrak{F}_\mathfrak{a}^d(U) \to \mathfrak{F}_\mathfrak{a}^d(L) \to \mathfrak{F}_\mathfrak{a}^d(L/U) \to 0.
\]

Since \(\dim(L/U) < d\) we have \(\mathfrak{F}_\mathfrak{a}^d(L/U) = 0\). On the other hand, by Definition 2.1 \(\dim(U/aU) < d\) and so \(\mathfrak{F}_\mathfrak{a}^d(U) = 0\). Thus the above long exact sequence implies that \(\mathfrak{F}_\mathfrak{a}^d(L) = 0\). Hence \(\dim(L/aL) < d\). Since \(U \subseteq L\), it follows from the maximality of \(U\) that \(U = L\).

ii) The short exact sequence

\[
0 \to U \to M \to M/U \to 0
\]

induces an exact sequence

\[
\cdots \to \mathfrak{F}_\mathfrak{a}^d(U) \to \mathfrak{F}_\mathfrak{a}^d(M) \to \mathfrak{F}_\mathfrak{a}^d(M/U) \to 0.
\]

Since \(\dim(U/aU) < d\), by definition 2.1 we have \(\mathfrak{F}_\mathfrak{a}^d(U) = 0\). So by using the above long exact sequence we conclude that \(\mathfrak{F}_\mathfrak{a}^d(M) \cong \mathfrak{F}_\mathfrak{a}^d(M/U)\). Therefore \(\text{Att}_R \mathfrak{F}_\mathfrak{a}^d(M) = \text{Att}_R \mathfrak{F}_\mathfrak{a}^d(M/U) \subseteq \text{Ass} M/U\) by [5, Theorem 3.1].

Now we show that \(\text{Ass} M/U \subseteq \text{Att}_R \mathfrak{F}_\mathfrak{a}^d(M) = \text{Att}_R \mathfrak{F}_\mathfrak{a}^d(M/U)\). Note that by (i) \(\dim M/U = d\) and by [5, Theorem 3.1] \(\text{Att}_R \mathfrak{F}_\mathfrak{a}^d(M/U) = \{ \mathfrak{p} \in \mathfrak{P} : \}

\[
\cdots \to \mathfrak{F}_\mathfrak{a}^d(U) \to \mathfrak{F}_\mathfrak{a}^d(M) \to \mathfrak{F}_\mathfrak{a}^d(M/U) \to 0.
\]
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\[ \text{Ass}_R M/U : \dim R/\mathfrak{p} = d \text{ and } \mathfrak{p} \supseteq \mathfrak{a} \} \]

If \( \mathfrak{p} \in \text{Ass} M/U \) then there exists a submodule \( K \) of \( M \) such that \( U \subsetneq K \leq M \) and \( R/\mathfrak{p} \simeq K/U \leq M/U \). By (i) \( \dim R/\mathfrak{p} = d \) and so it suffices to show that \( \mathfrak{a} \subseteq \mathfrak{p} \). If not, \( \dim R/(\mathfrak{a} + \mathfrak{p}) < \dim R/\mathfrak{p} = d \). Thus \( \dim((K/U)/\mathfrak{a}(K/U)) = \dim((R/\mathfrak{p})/\mathfrak{a}(R/\mathfrak{p})) = \dim(R/(\mathfrak{a} + \mathfrak{p})) < d \). Hence \( \mathfrak{F}_a^d(K/U) = 0 \). But the exact sequence

\[ 0 \rightarrow U \rightarrow K \rightarrow K/U \rightarrow 0 \]

induces an exact sequence

\[ \cdots \rightarrow \mathfrak{F}_a^d(U) \rightarrow \mathfrak{F}_a^d(K) \rightarrow \mathfrak{F}_a^d(K/U) \rightarrow 0. \]

Since \( \mathfrak{F}_a^d(U) = \mathfrak{F}_a^d(K/U) = 0 \) by the above long exact sequence we have \( \mathfrak{F}_a^d(K) = 0 \). Thus \( \dim(K/\mathfrak{a}K) < d \). But \( U \subsetneq K \) and so from the maximality of \( U \) we get a contradiction. Therefore \( \mathfrak{a} \subseteq \mathfrak{p} \) and the proof is complete.

iii) Let \( \mathfrak{p} \in \text{Ass}_R U \). Then there exists a submodule \( L \) of \( U \) such that \( R/\mathfrak{p} \simeq L \leq U \). Thus

\[ \dim R/(\mathfrak{a} + \mathfrak{p}) = \dim((R/\mathfrak{p})/\mathfrak{a}(R/\mathfrak{p})) \leq \dim(U/\mathfrak{a}U) < \dim(M/\mathfrak{a}M) = d. \]

Now, if \( \mathfrak{p} \in \text{Att}_R \mathfrak{F}_a^d(M) \) then \( \mathfrak{a} \subseteq \mathfrak{p} \) and \( \dim R/\mathfrak{p} = d \). Hence \( \dim R/(\mathfrak{a} + \mathfrak{p}) = d \) which is a contradiction. Therefore \( \text{Ass}_R U \subseteq \text{Ass}_R M - \text{Att}_R \mathfrak{F}_a^d(M) \). On the other hand,

\[ \text{Ass}_R M - \text{Att}_R \mathfrak{F}_a^d(M) \subseteq \text{Ass}_R M \subseteq \text{Ass}_R U \cup \text{Ass}_R M/U. \]

But by (ii) \( \text{Ass}_R M/U = \text{Att}_R \mathfrak{F}_a^d(M) \). Thus \( \text{Ass}_R M - \text{Att}_R \mathfrak{F}_a^d(M) \subseteq \text{Ass}_R U \). Therefore \( \text{Ass}_R M - \text{Att}_R \mathfrak{F}_a^d(M) = \text{Ass}_R U \).

iv) Since \( \text{Att}_R \mathfrak{F}_a^d(M) \subseteq V(\mathfrak{a}) \), it follows that \( \text{Ass}(M/U) \subseteq V(\mathfrak{a}) \) by (ii). Thus \( \mathfrak{a} \subseteq \cap_{\mathfrak{p} \in \text{Ass}(M/U)} \mathfrak{p} = \sqrt{(0 : (M/U))} \). This yields that \( M/U \) is an \( \mathfrak{a} \)-torsion \( R \)-module. Hence by [5, Lemma 2.1], \( \mathfrak{F}_a^d(M/U) \cong H^d_m(M/U) \). But in the proof of (ii) we saw that \( \mathfrak{F}_a^d(M/U) \cong \mathfrak{F}_a^d(M) \). Therefore \( \mathfrak{F}_a^d(M) \cong H^d_m(M/U) \).

Now we can prove the following main result.

**Theorem 2.3** Let \( \mathfrak{a} \) be an ideal of a local ring \((R, \mathfrak{m})\) and \( M \) a finitely generated \( R \)-module of dimension \( d \) such that \( \dim(M/\mathfrak{a}M) = d \). Then
Proof. Let $U := U_R(a, M)$. By Lemma 2.2 (iv), $\mathfrak{F}^d_a(M) \cong H^d_m(M/U)$. Thus $\text{Ann}_R(\mathfrak{F}^d_a(M)) = \text{Ann}_R(H^d_m(M/U))$. But by Theorem 1.1 we have

$$\text{Ann}_R(H^d_m(M/U)) = \text{Ann}_R((M/U)/T_R(m, M/U)).$$

Since $T_R(m, M/U) = 0$ by Lemma 2.2 (i), we conclude that

$$\text{Ann}_R(\mathfrak{F}^d_a(M)) = \text{Ann}_R(H^d_m(M/U)) = \text{Ann}_R M/U_R(a, M),$$

as required. □

**Proposition 2.4** Let $a$ be an ideal of a local ring $(R, m)$ and $M$ a finitely generated $R$-module of dimension $d$ such that $\dim(M/aM) = d$. Then

$$V(\text{Ann}_R \mathfrak{F}^d_a(M)) = \text{Supp}_R(M/U_R(a, M)).$$

Proof. By Theorem 2.3,

$$V(\text{Ann}_R \mathfrak{F}^d_a(M)) = V(\text{Ann}_R M/U_R(a, M)) = \text{Supp}_R(M/U_R(a, M)),$$

as required. □

**Theorem 2.5** Let $a$ be an ideal of a complete local ring $(R, m)$ and $M$ a finitely generated $R$-module of dimension $d$ such that $\dim(M/aM) = d$. Then

$$\text{Att}_R \mathfrak{F}^d_a(M) = \text{Min} \text{Supp}_R(M/U_R(a, M)) = \text{Ass}_R M/U_R(a, M).$$

Proof. By [13, Theorem 2.11 (ii)] $\text{Att}_R \mathfrak{F}^d_a(M) = \text{Min} V(\text{Ann}_R \mathfrak{F}^d_a(M))$.

Now the result follows by Proposition 2.4 and Lemma 2.2 (ii). □

The next Theorem gives us a description of $U_R(a, M)$.

**Theorem 2.6** Let $a$ be an ideal of a local ring $(R, m)$ and $M$ a finitely generated $R$-module of dimension $d$ such that $\dim(M/aM) = d$. Then

$$U_R(a, M) = \bigcap_{p_j \in \text{Ass}_R M \cap V(a)} N_{p_j}.$$
where $0 = \bigcap_{j=1}^{n} N_j$ denotes a reduced primary decomposition of the zero submodule $0$ in $M$ and $N_j$ is a $p_j$-primary submodule of $M$, for all $j = 1, \ldots, n$.

**Proof.** Set $N := \cap_{p_j \in \text{Ass}_{R} M \cap V(a)} N_j$. At first we show that $\dim(N/aN) < d$. By [14, Lemma 2.7] $\text{Ass}_{R} M/N = \text{Ass}_{R} M \cap V(a)$ and $\text{Ass}_{R} N = \text{Ass}_{R} M - \text{Ass}_{R} M \cap V(a)$. If $\dim N/aN = d$ then there exists a prime ideal $p \in \text{Supp}_{R} N \cap V(a)$ such that $\dim R/p = d$. Thus $p \in \text{Ass}_{R} M \cap V(a)$ and so $p \notin \text{Ass}_{R} N$. Since $p \in \text{Supp}_{R} N$ and $\dim R/p = d$ we have $p \in \text{Ass}_{R} N$ which is a contradiction. Therefore $\dim(N/aN) < d$ and so $N \subseteq U_{R}(a, M)$ by Definition 2.1.

Now we prove the reverse inclusion. To do this, suppose that there exists $x \in U$ such that $x \notin N$. Thus there exists an integer $t \in \{1, \ldots, n\}$ such that $x \notin N_t$ and $p_t \in \text{Ass}_{R} M \cap V(a)$. On the other hand, there exists an integer $k$ such that $(\sqrt{\text{Ann}_{R} Rx})^k x = 0$. Thus $(\sqrt{\text{Ann}_{R} Rx})^k x \subseteq N_t$. Since $x \notin N_t$ and $N_t$ is a $p_t$-primary submodule, it follows that $\cap_{p \in \text{Ass}_{R} Rx} p = \sqrt{\text{Ann}_{R} Rx} \subseteq p_t$. Thus there exists a prime ideal $p \in \text{Ass}_{R} Rx \subseteq \text{Ass}_{R} U$ such that $p \subseteq p_t$. Then, as $p \in \text{Ass}_{R} M$ and $\dim R/p_t = \dim M$ it follows that $p = p_t$. Hence $p \in \text{Ass}_{R} M \cap V(a) = \text{Att} \text{\zeta}^{d}_{a}(M)$. Now Lemma 2.2 (iii) implies that $p \notin \text{Ass}_{R} U$ which is a contradiction, because of $p \in \text{Ass}_{R} Rx \subseteq \text{Ass}_{R} U$. This completes the proof. □

**Corollary 2.7** Let $a$ be an ideal of a local ring $(R, m)$ and $M$ a finitely generated $R$-module of dimension $d$ such that $\dim(M/aM) = d$. Then

$$\text{Ann}_{R}(\text{\zeta}^{d}_{a}(M)) = \text{Ann}_{R}(M/ \cap_{p_j \in \text{Ass}_{R} M \cap V(a)} N_j),$$

where $0 = \bigcap_{j=1}^{n} N_j$ denotes a reduced primary decomposition of the zero submodule $0$ in $M$ and $N_j$ is a $p_j$-primary submodule of $M$, for all $j = 1, \ldots, n$.

**Proof.** The result follows from Theorems 2.3 and 2.6. □

3. The radical of the annihilators of the top formal local cohomology modules

Let $a$ be an ideal of a local ring $(R, m)$ and $M$ a finitely generated $R$-module of dimension $d$ such that $\dim(M/aM) = d$. The aim of this section will be to determine the radical of $\text{Ann}_{R}(\text{\zeta}^{d}_{a}(M))$. 
Definition 3.1 Let $M$ be a non-zero finitely generated $R$-module of finite dimension. We denote by $G_R(a, M)$ the largest submodule of $M$ such that $\text{Ass}_R(M) \cap V(a) \subseteq \text{Ass}_R(M/G_R(a, M))$.

Lemma 3.2 Let $(R, m)$ be a local ring and $a$ an ideal of $R$. Let $M$ be a finitely generated $R$-module of finite dimension $d$ such that $\dim(M/aM) = d$. Then $\dim(M/G_R(a, M)) = d$.

Proof. Since $\dim(M/aM) = d$ we have $\mathfrak{F}_a^d(M) \neq 0$. Thus $\text{Att}_R(\mathfrak{F}_a^d(M)) = \text{Ass}_R M \cap V(a) \neq \phi$.

Let $p \in \text{Ass}_R M \cap V(a)$. Then $p \in \text{Ass}_R(M/G_R(a, M))$. Thus $\text{Supp}_R(R/p) \subseteq \text{Supp}_R(M/G_R(a, M))$ and so $d = \dim(R/p) \leq \dim(M/G_R(a, M))$. On the other hand, $\dim(M/G_R(a, M)) \leq \dim M = d$. Therefore $d = \dim(M/G_R(a, M))$, as required.

Lemma 3.3 Let $a$ be an ideal of a local ring $(R, m)$ and $M$ a finitely generated $R$-module of dimension $d$ such that $\dim(M/aM) = d$. Then

$$U_R(a, M/G_R(a, M)) = 0.$$ 

Proof. Let $G := G_R(a, M)$. It suffices to show that for any non-zero submodule $L/G$ of $M/G$ we have $\dim((L/G)/a(L/G)) = \dim((M/G)/a(M/G))$. It is easy to see that $\text{Ass}_R(M) \cap V(a) \subseteq \text{Ass}_R(M/G) \subseteq \text{Ass}_R L/G \cup \text{Ass}_R M/L$. If $\text{Ass}_R(M) \cap V(a) \subseteq \text{Ass}_R(M/L)$ then since $G \subseteq L$ from the maximality of $G$ we get a contradiction. Thus there exists a prime ideal $p \in \text{Ass}_R(M) \cap V(a)$ such that $p \in \text{Ass}_R L/G$. Hence

$$\dim((R/p)/a(R/p)) \leq \dim((L/G)/a(L/G)) \leq \dim((M/G)/a(M/G))$$

$$\leq \dim(M/aM).$$

Since $p \in \text{Ass}_R M$, $\dim(R/p) = d$. Also, $p \in V(a)$ and so $\dim((R/p)/a(R/p)) = \dim(R/p) = d$. It follows that

$$d \leq \dim((L/G)/a(L/G)) \leq \dim((M/G)/a(M/G)) \leq d.$$ 

Therefore $\dim((L/G)/a(L/G)) = \dim((M/G)/a(M/G))$, as required.

Lemma 3.4 Let $a$ be an ideal of a local ring $(R, m)$ and $M$ a finitely generated $R$-module of dimension $d$ such that $\dim(M/aM) = d$. Then
Proof. Let $G := G_R(a, M)$. By definition 3.1 $Assh_R M \cap V(a) \subseteq Ass_R(M/G)$. Thus, by using Lemma 3.2 we conclude that

$$\{ p \in Ass_R M : \dim R/p = \dim M \} \cap V(a)$$

$$\subseteq \{ p \in Ass_R M/G : \dim R/p = \dim M/G \} \cap V(a)$$

and so $Att_R \mathfrak{F}^d_a(M) \subseteq Att_R \mathfrak{F}^d_a(M/G)$. On the other hand, the exact sequence

$$0 \to G \to M \to M/G \to 0$$

induces an exact sequence

$$\cdots \to \mathfrak{F}^d_a(G) \to \mathfrak{F}^d_a(M) \to \mathfrak{F}^d_a(M/G) \to 0.$$ 

Thus $Att_R(\mathfrak{F}^d_a(M/G)) \subseteq Att_R(\mathfrak{F}^d_a(M))$. Therefore $Att_R \mathfrak{F}^d_a(M) = Att_R \mathfrak{F}^d_a(M/G)$, the proof is complete. $\square$

**Lemma 3.5** Let $a$ be an ideal of a local ring $(R, m)$ and $M$ a finitely generated $R$-module of dimension $d$ such that $\dim(M/aM) = d$. Then

$$\sqrt{Ann_R(M/G_R(a, M))} = Ann_R(M/G_R(a, M)).$$

Proof. Let $G := G_R(a, M)$. Let $x \in \sqrt{Ann_R(M/G)}$. There exists an integer $n$ such that $x^n M \subseteq G$. Thus Lemma 3.4 implies that

$$Att_R(\mathfrak{F}^d_a(M)) = Att_R(\mathfrak{F}^d_a(M/G)) = Att_R(\mathfrak{F}^d_a(M/(x^n M + G))).$$

Since $Supp_R(M/(x^n M + G)) = Supp_R(M/(xM + G))$ by [5, Corollary 3.2] we have $Att_R(\mathfrak{F}^d_a(M/(x^n M + G))) = Att_R(\mathfrak{F}^d_a(M/(xM + G)))$. Hence

$$Att_R(\mathfrak{F}^d_a(M)) = Att_R(\mathfrak{F}^d_a(M/(xM + G))).$$

But $Att_R(\mathfrak{F}^d_a(M/(xM + G))) \subseteq Ass_R(M/(xM + G))$. Thus

$$Att_R(\mathfrak{F}^d_a(M)) = Assh_R M \cap V(a) \subseteq Ass_R(M/(xM + G)).$$
By definition of $G$ we conclude that $xM + G \subseteq G$. Therefore $xM \subseteq G$ and $x \in \text{Ann}_R(M/G)$, the proof is complete.

The following result is the main result of this section.

**Theorem 3.6** Let $a$ be an ideal of a local ring $(R, m)$ and $M$ a finitely generated $R$-module of dimension $d$ such that $\dim(M/aM) = d$. Then

$$\sqrt{\text{Ann}_R \mathfrak{F}^d_a(M)} = \text{Ann}_R M/G_R(a, M).$$

**Proof.** Let $G := G_R(a, M)$. By Lemma 3.4 and [6, 7.2.11] we have $\sqrt{\text{Ann}_R \mathfrak{F}^d_a(M)} = \sqrt{\text{Ann}_R \mathfrak{F}^d_a(M/G)}$. But by Lemma 3.2 $\dim(M/G) = d$ and so by Theorem 2.3 and Lemma 3.3,

$$\text{Ann}_R \mathfrak{F}^d_a(M/G) = \text{Ann}_R((M/G)/U_R(a, M/G)) = \text{Ann}_R M/G.$$

Now Lemma 3.5 implies that $\sqrt{\text{Ann}_R \mathfrak{F}^d_a(M/G)} = \sqrt{\text{Ann}_R M/G} = \text{Ann}_R M/G$. Thus $\sqrt{\text{Ann}_R \mathfrak{F}^d_a(M)} = \text{Ann}_R M/G$, as required. \qed

**Corollary 3.7** Let $a$ be an ideal of a local ring $(R, m)$ and $M$ a finitely generated $R$-module of dimension $d$ such that $\dim(M/aM) = \dim M$. Then

$$\cap_{p \in \text{Att}_R(\mathfrak{F}^d_a(M))} p = \text{Ann}_R M/G_R(a, M).$$

**Proof.** It follows by [6, 7.2.11] and Theorem 3.6. \qed

In the next result, we obtain a necessary and sufficient condition for the equality of the attached prime sets of the two top formal local cohomology modules.

**Proposition 3.8** Let $(R, m)$ be a local ring and $a$ an ideal of $R$. Let $M$ and $N$ be two finitely generated $R$-modules of dimension $d$ such that $\dim(M/aM) = \dim(N/aN) = d$. Then

$$\text{Att}_R \mathfrak{F}^d_a(M) = \text{Att}_R \mathfrak{F}^d_a(N) \text{ if and only if } \text{Supp}_R(M/G_R(a, M)) = \text{Supp}_R(N/G_R(a, N)).$$

**Proof.** If $\text{Att}_R \mathfrak{F}^d_a(M) = \text{Att}_R \mathfrak{F}^d_a(N)$ then $\text{Ann}_R M/G_R(a, M) = \text{Ann}_R N/G_R(a, N)$ by Corollary 3.7 and so $V(\text{Ann}_R(M/G_R(a, M))) = V(\text{Ann}_R(N/G_R(a, N)))$. \qed
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Thus \( \text{Supp}_R(M/G_R(a, M)) = \text{Supp}_R(N/G_R(a, N)) \).

Conversely, if \( \text{Supp}_R(M/G_R(a, M)) = \text{Supp}_R(N/G_R(a, N)) \) then by [5, Corollary 3.2] we have \( \text{Att}_R(\tilde{\mathfrak{a}}(M/G_R(a, M))) = \text{Att}_R(\tilde{\mathfrak{a}}(N/G_R(a, N))) \).

Therefore Lemma 3.4 implies that \( \text{Att}_R(\tilde{\mathfrak{a}}(M)) = \text{Att}_R(\tilde{\mathfrak{a}}(N)) \), as required. □

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References


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