A Lê-Greuel type formula for the image Milnor number

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Abstract. Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0) \) be a corank 1 finitely determined map germ. For a generic linear form \( p : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0) \) we denote by \( g : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^n, 0) \) the transverse slice of \( f \) with respect to \( p \). We prove that the sum of the image Milnor numbers \( \mu_I(f) + \mu_I(g) \) is equal to the number of critical points of \( p_{|X_s} : X_s \to \mathbb{C} \) on all the strata of \( X_s \), where \( X_s \) is the disentanglement of \( f \) (i.e., the image of a stabilisation \( f_s \) of \( f \)).

Key words: Image Milnor number, Lê-Greuel formula, finite determinacy.

1. Introduction

The Lê-Greuel formula [4], [6] provides a recursive method to compute the Milnor number of an isolated complete intersection singularity (ICIS). We recall that if \( (X, 0) \) is a \( d \)-dimensional ICIS defined as the zero locus of a map germ \( g : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-d}, 0) \), then the Milnor fibre \( X_s = g^{-1}(s) \) (where \( s \) is a generic value in \( \mathbb{C}^{n-d} \)) has the homotopy type of a bouquet of \( d \)-spheres and the number of such spheres is called the Milnor number \( \mu(X, 0) \). If \( d > 0 \), we can take \( p : \mathbb{C}^n \to \mathbb{C} \) a generic linear projection with \( H = p^{-1}(0) \) and such that \( (X \cap H, 0) \) is a \((d-1)\)-dimensional ICIS. Then,

\[
\mu(X, 0) + \mu(X \cap H, 0) = \dim \mathbb{C} \frac{\mathcal{O}_n}{(g) + J(g, p)},
\]

where \( \mathcal{O}_n \) is the ring of function germs from \( (\mathbb{C}^n, 0) \) to \( \mathbb{C} \), \((g)\) is the ideal in \( \mathcal{O}_n \) generated by the components of \( g \) and \( J(g, p) \) is the Jacobian ideal of \((g, p)\) (i.e., the ideal generated by the maximal minors of the Jacobian matrix). Note that \( X_s \) is smooth and if \( p \) is generic enough, then the re-

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striction \( p|_{X_s} : X_s \to \mathbb{C} \) is a Morse function and the dimension appearing in the right hand side of (1) is equal to the number of critical points of \( p|_{X_s} \).

The aim of this paper is to obtain a Lê-Greuel type formula for the image Milnor number of a finitely determined map germ \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0) \). Mond showed in [11] that the disentanglement \( X_s \) (i.e., the image of a stabilisation \( f_s \) of \( f \)) has the homotopy type of a bouquet of \( n \)-spheres and the number of such spheres is called the image Milnor number \( \mu_I(f, 0) \). The celebrated Mond’s conjecture says that

\[
\mathcal{A}_e\text{-codim}(f) \leq \mu_I(f),
\]

with equality if \( f \) is weighted homogeneous. Mond’s conjecture is known to be true for \( n = 1, 2 \) but it remains still open for \( n \geq 3 \) (see [11], [12]). We feel that our Lê-Greuel type formula can be useful to find a proof of the conjecture in the general case. In fact, it would be enough to prove that the module which controls the number of critical points of a generic linear function is Cohen-Macaulay and then, use an induction argument on the dimension \( n \) (see [1] for details about Mond’s conjecture).

We assume that \( f \) has corank 1 and \( n > 1 \). Then given a generic linear form \( p : \mathbb{C}^{n+1} \to \mathbb{C} \) we can see \( f \) as a 1-parameter unfolding of another map germ \( g : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^n, 0) \) which is the transverse slice of \( f \) with respect to \( p \). This means that \( g \) has image \( (X \cap H, 0) \), where \( (X, 0) \) is the image of \( f \) and \( H = p^{-1}(0) \). The disentanglement \( X_s \) is not smooth but it has a natural Whitney stratification given by the stable types. If \( p \) is generic enough, the restriction \( p|_{X_s} : X_s \to \mathbb{C} \) is a Morse function on each stratum. Our Lê-Greuel type formula is

\[
\mu_I(f) + \mu_I(g) = \# \Sigma(p|_{X_s}),
\]

where the right hand side of equation is the number of critical points of \( p|_{X_s} \) on all the strata of \( X_s \). The case \( n = 1 \) has to be considered separately, in this case we have

\[
\mu_I(f) + m_0(f) - 1 = \# \Sigma(p|_{X_s}),
\]

where \( m_0(f) \) is the multiplicity of the curve parametrized by \( f \). This makes sense, since \( \mu(X, 0) = m_0(X, 0) - 1 \) for a 0-dimensional ICIS \( (X, 0) \).
2. Multiple point spaces and Marar’s formula

In this section we recall Marar’s formula for the Euler characteristic of the disentanglement of a corank 1 finitely determined map germ. We first recall the Marar-Mond [9] construction of the $k$th-multiple point spaces for corank 1 map germs, which is based on the iterated divided differences. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a corank 1 map germ. We can choose coordinates in the source and target such that $f$ is written in the following form:

$$f(x, z) = (x, f_n(x, z), \ldots, f_p(x, z)), \quad x \in \mathbb{C}^{n-1}, \ z \in \mathbb{C}.$$

This forces that if $f(x_1, z_1) = f(x_2, z_2)$ then necessarily $x_1 = x_2$. Thus, it makes sense to embed the double point space of $f$ in $\mathbb{C}^{n-1} \times \mathbb{C}^2$ instead of $\mathbb{C}^n \times \mathbb{C}^n$. Analogously, we will consider the $k$th-multiple point space embedded in $\mathbb{C}^{n-1} \times \mathbb{C}^k$.

We construct an ideal $I_k(f) \subset \mathcal{O}_{n+k-1}$ defined as follows: $I_k(f)$ is generated by $(k-1)(p-n+1)$ functions $\Delta_i^{(j)} \in \mathcal{O}_{n+k-1}$, $1 \leq i \leq k-1$, $n \leq j \leq p$. Each $\Delta_i^{(j)}$ is a function only of the variables $x, z_1, \ldots, z_{i+1}$ such that:

$$\Delta_i^{(j)}(x, z_1, z_2) = \frac{f_j(x, z_1) - f_j(x, z_2)}{z_1 - z_2},$$

and for $1 \leq i \leq k-2$,

$$\Delta_{i+1}^{(j)}(x, z_1, \ldots, z_{i+2}) = \frac{\Delta_i^{(j)}(x, z_1, \ldots, z_i, z_{i+1}) - \Delta_i^{(j)}(x, z_1, \ldots, z_i, z_{i+2})}{z_{i+1} - z_{i+2}}.$$

**Definition 2.1** The $k$th-multiple point space is $D^k(f) = V(I_k(f))$, the zero locus in $(\mathbb{C}^{n+k-1}, 0)$ of the ideal $I_k(f)$.

(We remark that the $k$th-multiple point space is denoted by $\tilde{D}^k(f)$ instead of $D^k(f)$ in [9]).

If $f$ is stable, then, set-theoretically, $D^k(f)$ is the Zariski closure of the set of points $(x, z_1, \ldots, z_k) \in \mathbb{C}^{n+k-1}$ such that:

$$f(x, z_1) = \cdots = f(x, z_k), \quad z_i \neq z_j, \quad \text{for} \ i \neq j,$$

(see [9], [13]). But, in general, this may be not true if $f$ is not stable. For
instance, consider the cusp $f : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ given by $f(z) = (z^2, z^3)$. Since $f$ is one-to-one, the closure of the double point set is empty, but

$$D^2(f) = V(z_1 + z_2, z_1^2 + z_1z_2 + z_2^2).$$

This example also shows that the $k$th-multiple point space may be non-reduced in general.

The main result of Marar-Mond in [9] is that the $k$th-multiple point spaces can be used to characterize the stability and the finite determinacy of $f$.

**Theorem 2.2** ([9, 2.12]) Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ ($n < p$) be a finitely determined map germ of corank 1. Then:

1. $f$ is stable if and only if $D^k(f)$ is smooth of dimension $p - k(p - n)$, or empty, for $k \geq 2$.
2. $f$ is finitely determined if and only if for each $k$ with $p - k(p - n) \geq 0$, $D^k(f)$ is either an ICIS of dimension $p - k(p - n)$ or empty, and if, for those $k$ such that $p - k(p - n) < 0$, $D^k(f)$ consists at most of the point $\{0\}$.

The following construction is also due to Marar-Mond [9] and gives a refinement of the types of multiple points.

**Definition 2.3** Let $P = (r_1, \ldots, r_m)$ be a partition of $k$ (that is, $r_1 + \cdots + r_m = k$, with $r_1 \geq \cdots \geq r_m$). Let $I(P)$ be the ideal in $\mathcal{O}_{n-1+k}$ generated by the $k - m$ elements $z_i - z_{i+1}$ for $r_1 + \cdots + r_{j-1} + 1 \leq i \leq r_1 + \cdots + r_j$ for $j = 1, \ldots, m$. Define the ideal $I_k(f, P) = I_k(f) + I(P)$ and the $k$-multiple point space of $f$ with respect to the partition $P$ as $D^k(f, P) = V(I_k(f, P))$.

**Definition 2.4** We define a generic point of $D^k(f, P)$ as a point

$$(x, z_1, \ldots, z_1, \ldots, z_m, \ldots, z_m),$$

($z_i$ iterated $r_i$ times, and $z_i \neq z_j$ if $i \neq j$) such that the local algebra of $f$ at $(x, z_i)$ is isomorphic to $\mathbb{C}[t]/(t^{r_i})$, and such that

$$f(x, z_1) = \cdots = f(x, z_m).$$

If $f$ is stable, then $D^k(f, P)$ is equal to the Zariski closure of its generic
points (see [9]). Moreover, we have the following corollary, which extends Theorem 2.2 to the multiple point spaces with respect to the partitions.

**Corollary 2.5** ([9, 2.15]) If $f$ is finitely determined (resp. stable), then for each partition $P = (r_1, \ldots, r_m)$ of $k$ satisfying $p - k(p - n + 1) + m \geq 0$, the germ of $D^k(f, P)$ at $\{0\}$ is either an ICIS (resp. smooth) of dimension $p - k(p - n + 1) + m$, or empty. Moreover, those $D^k(f, P)$ for $P$ not satisfying the inequality consist at most of the single point $\{0\}$.

Let $f : (\mathbb{C}^p, 0) \rightarrow (\mathbb{C}^p, 0)$ be a finitely determined map germ of corank 1 and let $f_s : U_s \rightarrow X_s$ be a stabilization of $f$. For a partition $P$ of $k$, we denote by $\rho_P$ the mapping given as the composition of the inclusion $D^k(f_s, P) \hookrightarrow D^k(f_s)$, the projection $D^k(f_s) \rightarrow U_s$ and $f_s$. The following two results will be useful in the next section.

**Remark 2.6** ([8]) Let $P = (a_1, \ldots, a_h)$ be a partition of $k$, with $a_i \geq a_{i+1}$. If $y$ is a generic point of $D^k(f_s, P')$, where $P' = (b_1, \ldots, b_q)$, with $b_i \geq b_{i+1}$ and $P < P'$ then $\# \rho_P^{-1}(\rho_P(y))$ is the coefficient of the monomial $x_1^{b_1}x_2^{b_2} \ldots x_q^{b_q}$ in the polynomial $\prod_{i \geq 1}(x_1^{a_i} + x_2^{a_i} + \cdots x_q^{a_i})$.

**Lemma 2.7** ([7]) Let $h_k$ be the $k$-th complete symmetric function in variables $x_1, \ldots, x_q$, i.e., $h_k$ is the sum of all monomials of degree $k$ in the variables $x_1, \ldots, x_q$. Then

$$h_k = \sum_P \frac{1}{\prod_{i \geq 1} \alpha_i! \alpha_i} \prod_{i \geq 1} (x_1^{a_i} + \cdots + x_q^{a_i})^{\alpha_i},$$

where $P$ runs through the set of all ordered partitions of $k$.

The next step is to observe that the $k$th-multiple point space $D^k(f)$ is invariant under the action of the $k$th symmetric group $S_k$.

**Definition 2.8** Let $M$ be a $\mathbb{Q}$-vector space upon which $S_k$ acts. Then the **alternating part** of $M$, denoted by $\text{Alt}_k M$, is defined to be

$$\text{Alt}_k M := \{ m \in M : \sigma(m) = \text{sign}(\sigma)m, \text{ for all } \sigma \in S_k \}.$$ 

Given a topological space $X$ on which $S_k$ acts, the **alternating Euler characteristic** is
The following theorem of Goryunov-Mond in [3] allows us to compute the image Milnor number of \( f \) by means of a spectral sequence associated to the multiple point spaces.

**Theorem 2.9 ([3, 2.6])** Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0) \) be a corank 1 map germ and \( f_s \) a stabilisation of \( f \), for \( s \neq 0 \) and \( X_s \) the image of \( f_s \). Then,

\[
H_n(X_s, \mathbb{Q}) \cong \bigoplus_{k=2}^{n+1} \text{Alt}_k(H_{n-k+1}(D^k(f_s), \mathbb{Q})).
\]

Note that since \( X_s \) has the homotopy type of a wedge of \( n \)-spheres, the image Milnor number of \( f \) is the rank of \( H_n(X_s, \mathbb{Q}) \). If we consider \( H_n(X_s, \mathbb{Q}) \) as a \( \mathbb{Q} \)-vector space,

\[
\mu_1(f) = \dim_{\mathbb{Q}} H_n(X_s, \mathbb{Q}).
\]

So, by Theorem 2.9, the image Milnor number is

\[
\mu_1(f) = \sum_{k=2}^{n+1} \dim_{\mathbb{Q}} \text{Alt}_k(H_{n-k+1}(D^k(f_s), \mathbb{Q})).
\]

By [5, Corollary 2.8], we can compute the alternating Euler characteristic of \( D^k(f_s) \) as follows: for each partition \( P = (r_1, \ldots, r_s) \), we set

\[
\beta(P) = \frac{\text{sign}(P)}{\prod_i r_{\alpha_i}!},
\]

where \( \alpha_i := \# \{ j : r_j = i \} \) and \( \text{sign}(P) \) is the number \((-1)^{k-\sum \alpha_i}\). Then,

\[
\chi^{alt}(D^k(f_s)) = \sum_{|P|=k} \beta(P) \chi(D^k(f_s, P)).
\]

Moreover, by Theorem 2.2 and Corollary 2.5, \( D^k(f_s) \) (resp. \( D^k(f_s, P) \)) is a Milnor fibre of the ICIS \( D^k(f) \) (resp. \( D^k(f, P) \)), and hence it has the homotopy type of a wedge of spheres of real dimension \( \dim D^k(f) = n-k+1 \) (resp. \( \dim D^k(f, P) \)). Thus,
$$\dim_Q \text{Alt}_k(H_{n-k+1}(D^k(f_s), \mathbb{Q})) = (-1)^{n-k+1} \chi^\text{alt}(D^k(f_s)),$$

and

$$\chi(D^k(f_s, \mathcal{P})) = 1 + (-1)^{\dim D^k(f, \mathcal{P})} \mu(D^k(f, \mathcal{P})).$$

This gives the following version of Marar’s formula [8] in terms of the Milnor numbers of the multiple point spaces:

$$\mu_1(f) = \sum_{k=2}^{n+1} (-1)^{n-k+1} \sum_{|\mathcal{P}|=k} \beta(\mathcal{P}) (1 + (-1)^{\dim D^k(f, \mathcal{P})} \mu(D^k(f, \mathcal{P}))), \quad (4)$$

where the coefficients $\beta(\mathcal{P}) = 0$ when the sets $D^k(f, \mathcal{P})$ are empty, for $k = 2, \ldots, n+1$.

3. Lé-Greuel type formula

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a corank 1 finitely determined map germ. Let $p : \mathbb{C}^{n+1} \to \mathbb{C}$ be a generic linear projection such that $H = p^{-1}(0)$ is a generic hyperplane through the origin in $\mathbb{C}^{n+1}$. We can choose linear coordinates in $\mathbb{C}^{n+1}$ such that $p(y_1, \ldots, y_{n+1}) = y_1$. Then, we choose the coordinates in $\mathbb{C}^n$ in such a way that $f$ is written in the form

$$f(x_1, \ldots, x_{n-1}, z) = (x_1, \ldots, x_{n-1}, h_1(x_1, \ldots, x_{n-1}, z), h_2(x_1, \ldots, x_{n-1}, z)),$$

for some holomorphic functions $h_1, h_2$. We see $f$ as a 1-parameter unfolding of the map germ $g : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^n, 0)$ given by

$$g(x_2, \ldots, x_{n-1}, z) = (x_2, \ldots, x_{n-1}, h_1(0, x_2, \ldots, x_{n-1}, z), h_2(0, x_2, \ldots, x_{n-1}, z)).$$

We say that $g$ is the transverse slice of $f$ with respect to the generic hyperplane $H$. If $f$ has image $(X, 0)$ in $(\mathbb{C}^{n+1}, 0)$, then the image of $g$ in $(\mathbb{C}^n, 0)$ is isomorphic to $(X \cap H, 0)$.

We take $f_s$ a stabilisation of $f$ and denote by $X_s$ the image of $f_s$ (see [11] for the definition of stabilisation). Since $f$ has corank 1, $X_s$ has a natural Whitney stratification given by the stable types of $f_s$. In fact, the strata are the submanifolds
$$M^k(f_s, \mathcal{P}) := \epsilon^k(D^k(f_s, \mathcal{P})^0) \setminus \epsilon^{k+1}(D^{k+1}(f_s)),$$

where $D^k(f_s, \mathcal{P})^0$ is the set of generic points of $D^k(f_s, \mathcal{P})$, $\epsilon^k : \mathbb{C}^{n+k-1} \to \mathbb{C}^{n+1}$ is the map $(x, z_1, \ldots, z_k) \mapsto f_s(x, z_1)$ and $\mathcal{P}$ runs through all the partitions of $k$ with $k = 2, \ldots, n+1$. We can choose the generic linear projection $p : \mathbb{C}^{n+1} \to \mathbb{C}$ in such a way that the restriction to each stratum $M^k(f_s, \mathcal{P})$ is a Morse function. In other words, such that the restriction $p|_{X_s} : X_s \to \mathbb{C}$ is a Morse function on each stratum (this is one of the condition of be a stratified Morse function in the sense of [2]). We will denote by $\#(p|_{X_s})$ the number of critical points on all the strata of $X_s$.

Our first result in this section is for the case of a plane curve.

**Theorem 3.1** Let $f : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ be an injective map germ. Let $p : \mathbb{C}^2 \to \mathbb{C}$ be a generic linear projection, then

$$\#(p|_{X_s}) = \mu_I(f) + m_0(f) - 1,$$

where $m_0(f)$ is the multiplicity of $f$.

**Proof.** After a change of coordinates, we can assume that

$$f(t) = (t^k, c_m t^m + c_{m+1} t^{m+1} + \cdots),$$

where $k = m_0(f)$, $m > k$ and $c_m \neq 0$. The stabilisation $f_s$ is an immersion with only transverse double points. So, its image $X_s$ has only two strata: $M^2(f_s, (1, 1))$ is a 0-dimensional stratum composed by the transverse double points and $M^1(f_s, (1))$ is a 1-dimensional stratum given by the smooth points of $X_s$. Note that the number of double points of $f_s$ is the delta invariant of the plane curve, $\delta(X, 0)$, which is equal to $\mu_I(f)$ by [12, Theorem 2.3].

Let $p : \mathbb{C}^2 \to \mathbb{C}$ be a generic linear projection such that $p|_{X_s}$ is a Morse function on each stratum. Then:

$$\#(p|_{X_s}) = \#M^2(f_s, (1, 1)) + \#(p|M^1(f_s, (1))) = \mu_I(f) + \#(p|M^1(f_s, (1))).$$

Since $f_s$ is a local diffeomorphism on the stratum $M^1(f_s, (1))$, the number of critical points of $p|M^1(f_s, (1))$ is equal to the number of critical points of $p \circ f_s$ (here the points of $M^2(f_s, (1, 1))$ can be excluded by the genericity of $p$). Assume that $p(x, y) = Ax + By$ with $A \neq 0$. Then $p \circ f_s$ is a Morsification
of the function
\[ p \circ f(t) = At^k + B(c_m t^m + c_{m+1} t^{m+1} + \cdots) \]
The number of critical points of \( p \circ f \) is equal to \( \mu(p \circ f) = k-1 = m_0(f) - 1 \), which proves our formula. □

Next, we state and prove the formula for the case \( n > 1 \).

**Theorem 3.2** Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0) \) be a corank 1 finitely determined map germ with \( n > 1 \). Let \( p : \mathbb{C}^{n+1} \to \mathbb{C} \) be a generic linear projection which defines a transverse slice \( g : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^n, 0) \). Then,

\[ \#(p_j X_s) = \mu_1(f) + \mu_1(g) \]

**Proof.** By Marar’s formula (4):

\[
\mu_1(f) + \mu_1(g) = \sum_{k=2}^{n+1} (-1)^{n-k+1} \sum_{|\mathcal{P}|=k} \beta(\mathcal{P})(1 + (-1)^{\dim D^k(f, \mathcal{P})}\mu(D^k(f, \mathcal{P}))) \\
+ \sum_{k=2}^{n} (-1)^{n-k} \sum_{|\mathcal{P}|=k} \beta(\mathcal{P})(1 + (-1)^{\dim D^k(g, \mathcal{P})}\mu(D^k(g, \mathcal{P})))
\]

Note that if \( \dim D^k(f, \mathcal{P}) > 0 \), then \( \dim D^k(f, \mathcal{P}) = 1 + \dim D^k(g, \mathcal{P}) \). Moreover, if \( \dim D^k(f, \mathcal{P}) = 0 \), then \( D^k(g, \mathcal{P}) = \emptyset \). So, we can separate the formula into two parts, the first one for partitions with \( \dim D^k(f, \mathcal{P}) = 0 \), the second one for partitions with \( \dim D^k(f, \mathcal{P}) > 0 \). Thus,

\[
\mu_1(f) + \mu_1(g) = \sum_{k=2}^{n+1} (-1)^{n+k-1} \sum_{|\mathcal{P}|=k} \beta(\mathcal{P})(1 + \mu(D^k(f, \mathcal{P}))) \\
+ \sum_{k=2}^{n} (-1)^{n+k-1} \sum_{|\mathcal{P}|=k} \beta(\mathcal{P})(1 + \mu(D^k(g, \mathcal{P})))
\]

\times \beta(\mathcal{P})(-1)^{\dim D^k(f, \mathcal{P})}(\mu(D^k(f, \mathcal{P}))) + \mu(D^k(g, \mathcal{P}))
\]

If \( \dim D^k(f, \mathcal{P}) = 0 \), the Milnor number of \( D^k(f, \mathcal{P}) \) is
\[ \mu(D^k(f, \mathcal{P})) = \deg(D^k(f, \mathcal{P})) - 1, \]

where \( \deg \) is the degree of the map germ that defines the 0-dimensional ICIS \( D^k(f, \mathcal{P}) \). Note that we can see \( \deg(D^k(f, \mathcal{P})) \) as the number of critical points of \( \tilde{p}|_{D^k(f_s, \mathcal{P})} \).

We choose the coordinates such that \( p(y_1, \ldots, y_{n+1}) = y_1 \). We denote by \( \tilde{p} : \mathbb{C}^{n+k-1} \to \mathbb{C} \) the projection onto the first coordinate. Then:

\[ D^k(g, \mathcal{P}) = D^k(f, \mathcal{P}) \cap \tilde{p}^{-1}(0). \]

By the Lê-Greuel formula for ICIS [4], [6],

\[ \mu(D^k(f, \mathcal{P})) + \mu(D^k(g, \mathcal{P})) = \#(\tilde{p}|_{D^k(f_s, \mathcal{P})}). \]

It is easy to check that \((-1)^{\dim D^k(f)} \cdot \text{sign}(\mathcal{P})(-1)^{\dim D^k(f, \mathcal{P})} = 1\) for any partition \( \mathcal{P} \). Thus, we get:

\[ \mu_I(f) + \mu_I(g) = \sum_{k=2}^{n+1} \sum_{|\mathcal{P}|=k} \frac{\#(\tilde{p}|_{D^k(f_s, \mathcal{P})})}{\gamma(\mathcal{P})}, \]

where \( \gamma(\mathcal{P}) = \prod_i \alpha_i! i^{\alpha_i} \).

Let \( \mathcal{P} \) be a partition of \( k \), if \( |\mathcal{P}'| = k \) and \( \mathcal{P}' \geq \mathcal{P} \) then any critical point of \( \tilde{p}|_{D^k(f_s, \mathcal{P}')} \) is a critical point of \( \tilde{p}|_{D^k(f_s, \mathcal{P})} \). This implies

\[ \#(\tilde{p}|_{D^k(f_s, \mathcal{P})}) = \sum_{|\mathcal{P}'|=k, \mathcal{P}' \geq \mathcal{P}} \alpha(\mathcal{P}, \mathcal{P}') \#(\tilde{p}|_{D^k(f_s, \mathcal{P}')}^0), \]

where \( \alpha(\mathcal{P}, \mathcal{P}') \) is defined by

\[ \alpha(\mathcal{P}, \mathcal{P}') := \frac{\#(\rho_{\mathcal{P}'}^{-1}(\rho_{\mathcal{P}'}(y)))}{\#(\rho_{\mathcal{P}'}^{-1}(\rho_{\mathcal{P}'}(y)))} \]

for a generic point \( y \) in \( D^k(f_s, \mathcal{P}') \). We can see \( \alpha(\mathcal{P}, \mathcal{P}') \) as the number of times that a generic point of \( D^k(f_s, \mathcal{P}') \) appears repeated in \( D^k(f_s, \mathcal{P}) \). By Remark 2.6 and Lemma 2.7,
\[
\mu(f) + \mu(g) = \sum_{k=2}^{n+1} \sum_{|P|=k} \frac{\# \Sigma(p|D^k(f_s, P))}{\gamma(P)}
\]

\[
= \sum_{k=2}^{n+1} \sum_{|P|=k} \sum_{|P'|=k} \frac{\alpha(P, P')}{\gamma(P)} \# \Sigma(p|D^k(f_s, P'))
\]

\[
= \sum_{k=2}^{n+1} \sum_{|P'|=k} \left( \sum_{|P|=k} \frac{\# P^{-1}(P'(y))}{\gamma(P)} \right) \frac{\# \Sigma(p|D^k(f_s, P'))}{\# P^{-1}(P'(y))}
\]

\[
= \sum_{k=2}^{n+1} \sum_{|P'|=k} \# \Sigma(p|D^k(f_s, P'))
\]

which is nothing but the number of critical points of \(p|X_s\).

\[
\square
\]

4. Examples

In this section, we give some examples to illustrate the formulas of theorems 3.1 and 3.2.

**Example 4.1** (The singular plane curve \(E_6\)) Let \(f(z) = (z^3, z^4)\) be the singular plane curve \(E_6\), let \(f_s(z) = (z^3 + sz, z^4 + (5/4)sz^2)\) be a stabilisation of \(f\), for \(s \neq 0\).

Let \(M^2(f_s, (1, 1))\) be the 0-dimensional stratum of \(X_s\). It is composed by three points, they correspond to three double transversal points. Let \(M^1(f_s, (1))\) be the 1-dimensional stratum. If we compose \(f_s\) with \(p(z, u) = z\) there are two critical points in a neighbourhood of the origin, so \(\# \sum p|X_s = 5\).

![Figure 1. The curve \(E_6\) and its stabilisation for \(s < 0\).](image-url)
Now, since the multiplicity of $f$, $m_0(f) = 3$ and the image Milnor number of $f$ is $\mu_1(f) = 3$, $\mu_1(f) + m_0(f) - 1 = 5$ as predicted by the formula.

When $n > 1$, we proceed in the following way: Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$ be a corank 1 finitely determined map germ written as

$$f(x, z) = (x, h_1(x, z), h_2(x, z)), \quad x \in \mathbb{C}^{n-1}, \quad z \in \mathbb{C}.$$ 

Let $f_s$ be a stabilisation of $f$. The image of $f_s$ is denoted by $X_s$. First, we calculate the number of critical points of the restriction of $p$ to $X_s$, for the generic linear projection $p(y_1, \ldots, y_{n+1}) = y_1$. We separate the image set $X_s$ in strata of different dimensions given by stable types, which correspond to the sets $M^k(f_s, \mathcal{P})$. The $n$-dimensional stratum, $M^1(f_s, (1))$, is composed of the regular part of $f_s$. So, the restriction $p|_{M^1(f_s)}$ has not critical points.

The $(n-1)$-dimensional stratum is composed of $M^2(f_s, (1, 1))$. To calculate the critical points, we will work with the inverse image by $\epsilon^2$, that is, $D^2(f_s, (1, 1)) = D^2(f_s)$. The double point space $D^2(f_s)$ is a subset of $\mathbb{C}^{n+1}$, but we take a projection of $D^2(f_s)$ in the first $n$ variables. So, we denote by $D(f_s)$ the projection of double point space in $\mathbb{C}^n$. The double point space $D(f_s)$ is a hypersurface in $\mathbb{C}^n$ given by the resultant of $P_s$ and $Q_s$ with respect to $z_2$, where $P_s = (h_{1,s}(x, z_2) - h_{1,s}(x, z_1))/(z_2 - z_1)$ and $Q_s = (h_{2,s}(x, z_2) - h_{2,s}(x, z_1))/(z_2 - z_1)$. This gives the defining equation of $D(f_s)$, denoted by $\lambda_s(x, z) = 0$.

To calculate the critical points of the set $D(f_s)$ we take the linear projection $\tilde{p}(x_1, \ldots, x_{n-1}, z) = x_1$. Note that the hypersurface $D(f_s)$ also contains the critical points of the other $k$-dimensional strata, with $k < n - 1$. Then, it will be sufficient to compute critical points here, in order to have all the critical points. We have that $(x_1, \ldots, x_{n-1}, z)$ is a critical point of $\tilde{p}|_{D(f_s)}$ if $\lambda_s(x, z) = 0$ and $J(\lambda_s, \tilde{p})(x, z) = 0$, where $J(\lambda_s, \tilde{p})$ is the Jacobian determinant of $\lambda$ and $\tilde{p}$.
If a critical point of $\tilde{p}|_{D(f_s)}$ corresponds to a $m$-multiple point, then we will have $m$ critical points in $D(f_s)$ for one in the image of $f_s$. Thus, once the critical points of each type are obtained, we have to divide by the multiplicity of the point. In this way, we obtain the number of critical points of $p$ in the image of $f_s$.

On the other hand, we compute separately the image Milnor numbers of $f$ and $g$ in order to check the formulas.

**Example 4.2** (The germ $F_4$ in $\mathbb{C}^3$) Let $f(x, z) = (x, z^2, z^5 + x^3 z)$ be the germ $F_4$. Let $f_s(x, z) = (x, z^2, z^5 + xsz^3 + (x^3 - 5xs - s)z)$ be a stabilisation of $f$, for $s \neq 0$. By [10], $f$ is a 1-parameter unfolding of the plane curve $A_4$, $g(z) = (z^2, z^5)$ and in fact, $g$ is the transverse slice of $f$.

Figure 3. The germ $F_4$ and its stabilisation for $s > 0$.

Let $M^3(f_s, (1, 1, 1)) \cup M^2(f_s, (2))$ be the 0-dimensional strata of $X_s$. In our case, there are not triple points and there are three cross caps in $M^2(f_s, (2))$.

Let $M^2(f_s, (1, 1))$ be the 1-dimensional stratum of $X_s$. As we said, let $D^2(f_s)$ be the double point curve in $\mathbb{C}^3$ and by projecting in the first two coordinates, we have the double point curve in $\mathbb{C}^2$, denoted by $D(f_s)$.

We compute the resultant of $P_s$ and $Q_s$ respect to $z_2$, where $P_s$ and $Q_s$ are the divided differences. The double point curve of $f_s$ in $\mathbb{C}^2$ is the plane curve

$$\lambda_s(x, z) = -s - 5sx + x^3 + sxz^2 + z^4.$$

The critical points of the restriction $p|_{D(f_s)}$ are given by $\lambda_s(x_0, z_0) = 0$ and $J(\lambda_s, \tilde{p})(x_0, z_0) = 0$, where $\tilde{p}(x, z) = x$.

Nine critical points are obtained. Three of these points are cusps in $g_{x,s}$ which correspond to the three cross caps of $f_s$. Then, the other six critical points in $\tilde{p}|_{\lambda_s(x_0, z_0) = 0}$ correspond to three tacnodes in $g_{x,s}$ which are represented in the double point curve when a vertical line is tangent at two
points of $D(f_s)$. So, each two of these critical points in $\lambda_s$ correspond to one tacnode of $g_{x,s}$ in $M^2(f_s, (1,1))$. Note that in the Fig. 4 there are only two tacnodes, that is because the other is a complex tacnode.

Finally, in the 2-dimensional stratum $M^1(f_s, (1))$ there are not critical points. So, the number of critical points in $X_s$ is $\#\Sigma p|_{X_s} = 6$, three cusps, three tacnodes and zero triple points. Then, $\#\Sigma p|_{X_s} = C + J + T$ where $C, J, T$ are the numbers of cusps, tacnodes and triple points respectively of $g_{x,s}$. By [10], $\mu_1(f) = C + J + T - \delta(g)$. Since $g$ is a plane curve, we have that $\mu_1(g) = \delta(g)$ (see [12]). So,

$$\#\Sigma p|_{X_s} = C + J + T = \mu_1(f) + \mu_1(g).$$

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