The influence of nonnormal noncyclic subgroups on the structure of finite groups

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Abstract. We obtain a complete classification of finite groups in which all noncyclic proper subgroups are nonnormal, and we apply this classification to investigate some structures of finite groups.

Key words: noncyclic subgroup, nonnormal, nonabelian simple group.

1. Introduction

In this paper all groups are assumed to be finite. It is known that a group $G$ is called a Dedekind-group if all subgroups of $G$ are normal in $G$ (see [6, Theorem 5.3.7]), and a group $G$ is said to be a simple group if all nontrivial subgroups of $G$ are nonnormal in $G$. As generalizations, it is natural to investigate the normality of some particular subgroups. In [1], Buckley characterized groups in which all minimal subgroups are normal, such groups are called PN-groups. Note that a group in which all cyclic subgroups are normal is also a Dedekind-group. For the noncyclic subgroups, [2] and [5] classified all $p$-groups in which all noncyclic subgroups are normal. And in [3], Kutnar, Marušić and the authors classified noncyclic groups in which all supersolvable noncyclic subgroups are selfnormalizing.

As a further study of the normality of noncyclic subgroups, the main goal of this paper is to classify groups in which all noncyclic proper subgroups are nonnormal. For convenience, we call a group $G$ an NCNN-group if $G$ has at least one noncyclic proper subgroup and all noncyclic proper subgroups of $G$ are nonnormal in $G$.

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For NCNN-groups, we have the following result, the proof of which is given in Section 2.

**Theorem 1.1** A group $G$ is an NCNN-group if and only if one of the following statements holds:

(1) $G/\Phi(G)$ is a nonabelian simple group with $\Phi(G) = Z(G)$ being cyclic, where $\Phi(G)$ is the Frattini subgroup of $G$ and $Z(G)$ is the center of $G$;

(2) $G \cong \langle a, b \mid a^m = b^n = 1, b^{-1}ab = a^r \rangle$, where $m > r \geq 1$, $n \geq 1$ are positive integers and $q$ is the smallest prime divisor of $|G|$ such that $((r - 1)q, m) = 1$ and $r^q \equiv 1 \pmod{m}$.

Next we will apply Theorem 1.1 to investigate some structures of groups.

**Lemma 1.2** ([3, Theorem 1.2]) Let $G$ be a group having at least one noncyclic proper subgroup. Then all noncyclic proper subgroups of $G$ are selfnormalizing in $G$ if and only if $G \cong \langle a, b \mid a^m = b^n = 1, b^{-1}ab = a^r \rangle$, where $m > r \geq 1$, $n \geq 1$ are positive integers and $q$ is the smallest prime divisor of $|G|$ such that $((r - 1)q, m) = 1$ and $r^q \equiv 1 \pmod{m}$.

Combining Theorem 1.1 and Lemma 1.2 together, we obtain the following interesting result for noncyclic subgroups.

**Theorem 1.3** Let $G$ be a solvable group having at least one noncyclic proper subgroup. Then all noncyclic proper subgroups of $G$ are nonnormal in $G$ if and only if all noncyclic proper subgroups of $G$ are selfnormalizing in $G$.

The alternating group $A_5$ shows that Theorem 1.3 is not true if $G$ is a nonsolvable group.

Note that all PN-groups are solvable by [1]. Then we can easily get the following theorem by Theorem 1.1.

**Theorem 1.4** Let $G$ be a PN-group having at least one noncyclic proper subgroup. Then all noncyclic proper subgroups of $G$ are nonnormal in $G$ if and only if $G \cong \langle a, b \mid a^m = b^n = 1, b^{-1}ab = a^r \rangle$, where $m > r \geq 1$, $n \geq 2$ are positive integers and $q$ is the smallest prime divisor of $|G|$ such that $((r - 1)q, m) = 1$ and $r^q \equiv 1 \pmod{m}$.

The following three corollaries are direct consequences of Theorem 1.1.

**Corollary 1.5** Let $G$ be a group having at least one noncyclic proper
subgroup. Then all noncyclic proper subgroups of $G$ are not subnormal in $G$ if and only if one of the following statements holds:

1. $G/\Phi(G)$ is a nonabelian simple group with $\Phi(G) = Z(G)$ being cyclic;
2. $G \cong \langle a, b \mid a^m = b^n = 1, b^{-1}ab = a^r \rangle$, where $m > r \geq 1$, $n \geq 1$ are positive integers and $q$ is the smallest prime divisor of $|G|$ such that $((r - 1)q, m) = 1$ and $r^q \equiv 1 \pmod{m}$.

**Corollary 1.6** Let $G$ be a group having at least one noncyclic proper subgroup. Then for any noncyclic proper subgroup $H$ of $G$ we always have that $H_G$ (the largest normal subgroup of $G$ that is contained in $H$) is cyclic if and only if one of the following statements holds:

1. $G/\Phi(G)$ is a nonabelian simple group with $\Phi(G) = Z(G)$ being cyclic;
2. $G \cong \langle a, b \mid a^m = b^n = 1, b^{-1}ab = a^r \rangle$, where $m > r \geq 1$, $n \geq 1$ are positive integers and $q$ is the smallest prime divisor of $|G|$ such that $((r - 1)q, m) = 1$ and $r^q \equiv 1 \pmod{m}$.

**Corollary 1.7** Let $G$ be an NCNN-group. If $G$ is solvable, then $G$ is supersolvable.

Note that a group having at most three conjugacy classes of noncyclic proper subgroups is solvable by [4]. Combining Corollary 1.7 and [4] together, we have the following corollary.

**Corollary 1.8** Let $G$ be an NCNN-group. If $G$ has at most three conjugacy classes of noncyclic proper subgroups, then $G$ is supersolvable.

2. **Proof of Theorem 1.1**

*Proof.* (1) For the necessity part.

(i) Suppose that $G$ is nonsolvable. By [6, Exercise 10.5.7], we have that all maximal subgroups of $G$ are noncyclic. If $G$ is a nonabelian simple group, then $G$ clearly satisfies the hypothesis. Next we assume that $G$ is not a nonabelian simple group. Let $N$ be a maximal nontrivial normal subgroup of $G$. By the hypothesis, we have that $N$ is cyclic. Then $G/N$ must be a nonabelian simple group. We claim that

$$N \leq \Phi(G).$$
Otherwise, assume $N \not\in \Phi(G)$. Let $M$ be a maximal subgroup of $G$ such that $N \not\leq M$. Then $G = NM$. It is obvious that $N \cap M \leq M$. Moreover, $N \cap M \unlhd N$ since $N$ is cyclic. So $N \cap M \leq G$. We have $G/(N \cap M) = N/(N \cap M) \rtimes M/(N \cap M)$. Let $\bar{G} = G/(N \cap M)$, $\bar{N} = N/(N \cap M)$ and $\bar{M} = M/(N \cap M)$. It is obvious that $\bar{M} \cong G/N$ is a nonabelian simple group. By $N/C$-theorem, $\bar{G}/C_{\bar{G}}(\bar{N}) = N/\bar{N}(\bar{N}) \cong \text{Aut}(\bar{N})$. Since $\bar{N}$ is cyclic, we have that $\text{Aut}(\bar{N})$ is abelian. However, since $\bar{G}/C_{\bar{G}}(\bar{N}) \cong (\bar{G}/\bar{N})/(C_{\bar{G}}(\bar{N})/\bar{N})$ and $\bar{G}/\bar{N} \cong \bar{M}$ is a nonabelian simple group, it follows that $C_{\bar{G}}(\bar{N}) = \bar{G}$. That is, $\bar{N} \leq Z(\bar{G})$. Thus $\bar{M} \cong \bar{G}$. It implies that $M \unlhd G$. By the hypothesis, we have that $M$ is cyclic. Then it is easy to see that $G$ is solvable, a contradiction. Hence $N \leq \Phi(G)$.

Since $G/N$ is a nonabelian simple group, it follows that $N = \Phi(G)$. So $G/\Phi(G)$ is a nonabelian simple group, where $\Phi(G)$ is cyclic. Moreover, we can easily get $\Phi(G) = Z(G)$ by $N/C$-theorem.

(ii) Suppose that $G$ is solvable. If $G$ is nilpotent, then all maximal subgroups of $G$ are normal in $G$. By the hypothesis, we have that all maximal subgroups of $G$ are cyclic, this contradicts that $G$ has at least one noncyclic proper subgroup.

Thus $G$ is nonnilpotent. Since $G$ is solvable, one has that $G$ has a maximal subgroup $L$ such that $L \unlhd G$. By the hypothesis, we have that $L$ is cyclic. Assume $G/L \cong \mathbb{Z}_e$, where $e$ is a prime divisor of $|G|$. Let $E \in Syl_e(G)$. Then $G = LE$. Let $K$ be an $e'$-Hall subgroup of $L$. It is obvious that $K \unlhd G$ since $L$ is cyclic. Thus $G = K \rtimes E$. We claim that

$E$ is cyclic.

Otherwise, assume that $E$ is noncyclic. Let $E_1$ and $E_2$ be two distinct maximal subgroups of $E$. It is easy to see that $K \rtimes E_1$ and $K \rtimes E_2$ are normal in $K \rtimes E = G$. By the hypothesis, we have that $K \rtimes E_1$ and $K \rtimes E_2$ are cyclic. It follows that $E_1 \leq C_G(K)$ and $E_2 \leq C_G(K)$. So $E = E_1E_2 \leq C_G(K)$. It implies that $G$ is nilpotent, a contradiction. Hence $E$ is cyclic.

Thus $G$ is a group in which all Sylow subgroups are cyclic. By [6, Theorem 10.1.10], we have $G = \langle a, b \mid a^m = b^s = 1, b^{-1}ab = a^r \rangle$, where $m$ and $s$ are positive integers such that $((r - 1)s, m) = 1$ and $r^s \equiv 1 \pmod{m}$. We claim that

$s$ is a prime-power.
Otherwise, assume that $t_1$ and $t_2$ are two distinct prime divisors of $s$. Then $\langle b^{t_1} \rangle$ and $\langle b^{t_2} \rangle$ are two distinct maximal subgroups of $\langle b \rangle$. It is easy to see that $\langle a \rangle \rtimes \langle b^{t_1} \rangle$ and $\langle a \rangle \rtimes \langle b^{t_2} \rangle$ are normal in $\langle a \rangle \rtimes \langle b \rangle = G$. By the hypothesis, we have that $\langle a \rangle \rtimes \langle b^{t_1} \rangle$ and $\langle a \rangle \rtimes \langle b^{t_2} \rangle$ are cyclic. Then $\langle b^{t_1} \rangle \leq C_G(\langle a \rangle)$ and $\langle b^{t_2} \rangle \leq C_G(\langle a \rangle)$. Thus $\langle b \rangle = \langle b^{t_1} \rangle \langle b^{t_2} \rangle \leq C_G(\langle a \rangle)$. It follows that $G$ is cyclic, a contradiction. So $s$ is a prime-power.

Assume $s = q^n$, where $q$ is a prime and $n \geq 1$. Since $\langle a \rangle \rtimes \langle b^q \rangle$ is normal in $\langle a \rangle \rtimes \langle b \rangle = G$. By the hypothesis, we have that $\langle a \rangle \rtimes \langle b^q \rangle$ is cyclic. Thus $r^q \equiv 1 \pmod{m}$.

Next we claim that

$q$ is the smallest prime divisor of $|G|$.

Otherwise, let $f$ be the smallest prime divisor of $|G|$ and $f \neq q$. Let $F \in \text{Syl}_f(G)$. By above argument, $F$ is cyclic. Then $G$ is $f$-nilpotent by [6, Theorem 10.1.9]. That is, there exists a normal subgroup $T$ of $G$ such that $G = T \rtimes F$. By the hypothesis, $T$ is cyclic. Since $q \neq f$, we have $\langle b \rangle \leq T$. Thus $\langle b \rangle \leq G$. It follows that $G$ is cyclic, a contradiction. So $q$ is the smallest prime divisor of $|G|$.

(2) For the sufficiency part.

If $G/\Phi(G)$ is a nonabelian simple group with $\Phi(G) = Z(G)$ being cyclic, it is easy to show that all noncyclic proper subgroups of $G$ are nonnormal in $G$.

Next assume $G = \langle a, b \mid a^m = b^n = 1, b^{-1}ab = a^r \rangle$, where $m \geq 1$, $n \geq 1$ are positive integers and $q$ is the smallest prime divisor of $|G|$ such that $((r - 1)q, m) = 1$ and $r^q \equiv 1 \pmod{m}$. Let $R$ be a noncyclic proper subgroup of $G$. By the definition of $G$, it is easy to show that $R = \langle a^i \rangle \rtimes \langle b^x \rangle$ for some $x \in G$ and some positive integer $i$ such that $\langle a^i \rangle < \langle a \rangle$. If $R \leq G$, then $G/R = \langle a \rangle \rtimes \langle b^x \rangle / \langle a^i \rangle \rtimes \langle b^x \rangle$ is cyclic. It follows that $G' \leq R$. Since $b^{-1}ab = a^r$, one has $[a, b] = a^{-1}b^{-1}ab = a^{1-r}$. Thus $a^{1-r} \in R$. It follows that $\langle a^{r-1} \rangle \leq R$. Since $(r - 1, m) = 1$, we have $\langle a^{r-1} \rangle = \langle a \rangle$. Then $\langle a \rangle \leq R$, this contradicts that $\langle a^i \rangle < \langle a \rangle$. Hence all noncyclic proper subgroups of $G$ are nonnormal in $G$. □

References


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