An almost complex Castelnuovo de Franchis theorem

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Abstract. Given a compact almost complex manifold, we prove a Castelnuovo–de Franchis type theorem for it.

Key words: Almost complex structure, Castelnuovo–de Franchis theorem, Riemann surface, fundamental group.

1. Introduction

Given a smooth complex projective variety \( X \), the classical Castelnuovo-de Franchis theorem [2, 4] associates to an isotropic subspace of \( H^0(X, \Omega^1_X) \) of dimension greater than one an irrational pencil on \( X \). The topological nature of this theorem was brought out by Catanese [3], who established a bijective correspondence between subspaces \( \bar{U} \subset H^1(X, \mathbb{C}) \) of the form \( U \oplus \bar{U} \) with \( U \) being a maximal isotropic subspace of \( H^1(X, \mathbb{C}) \) of dimension \( g \geq 1 \) and irrational fibrations on \( X \) of genus \( g \). The purpose of this note is to emphasize this topological content further. We extract topological hypotheses that allow the theorem to go through when we have only an almost complex structure.

In what follows, \( M \) will be a compact smooth manifold of dimension \( 2k \). Let

\[
J_M : TM \rightarrow TM
\]

be an almost complex structure on \( M \), meaning \( J_M \circ J_M = -\text{Id}_{TM} \).

Definition 1.1 We shall say that a collection of closed complex 1–forms \( \omega_1, \ldots, \omega_n \) on \( M \) are in general position if

1. the zero-sets

\[
Z(\omega_i) := \{ x \in M \mid \omega_i(x) = 0 \} \subset M
\]

are smooth embedded submanifolds, and

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these submanifolds \(Z(\omega_i)\) intersect transversally.

We are now in a position to state the main theorem of this note.

**Theorem 1.2** Let \(M\) be a compact smooth \(2k\)-manifold equipped with an almost complex structure \(J_M\). Let \(\omega_1, \ldots, \omega_g\) be closed complex 1-forms on \(M\) linearly independent over \(\mathbb{C}\), with \(g \geq 2\), such that

- each \(\omega_i\) is of type \((1,0)\), meaning \(\omega_i(J_M(v)) = \sqrt{-1} \cdot \omega_i(v)\) for all \(v \in TM\), and
- \(\omega_i\) are in general position with \(\omega_i \wedge \omega_j = 0\) for all \(i, j\).

Then there exists a smooth almost holomorphic map \(f : M \rightarrow C\) to a compact Riemann surface of genus at least \(g\), and there are linearly independent holomorphic 1-forms \(\eta_1, \ldots, \eta_g\) on \(C\), such that \(\omega_i = f^* \eta_i\) for all \(i\).

2. Leaf space and almost complex blow-up

2.1. Leaf space

Assume that \(\omega_i\) are forms as in Theorem 1.2. Since \(\omega_i \wedge \omega_j = 0\) for all \(1 \leq i, j \leq g\), it follows that there are complex valued smooth functions \(f_{i,j}\) such that

\[
\omega_i = f_{i,j} \omega_j
\]

(1)

wherever \(\omega_j \neq 0\). Hence the collection

\[W = \{\omega_1, \ldots, \omega_g\}\]

determines a complex line subbundle of the complexified cotangent bundle \((T^*M) \otimes \mathbb{C}\) over the open subset

\[V := M \setminus \bigcap_{i=1}^g Z(\omega_i) \subset M.\]

**Lemma 2.1** Let \(\mathcal{F} := \{v \in TV \mid \omega_i(v) = 0 \quad \forall \ 1 \leq i \leq g\} \subset TV\) be the distribution on \(V\) defined by \(W\). Then \(\mathcal{F}\) is integrable and defines a foliation of real codimension two on \(V\).

**Proof.** For any \(x \in M\) and \(1 \leq i \leq g\), consider the \(\mathbb{R}\)-linear homomorphism
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Since \( \omega_i \) is of type \((1,0)\), if \( \omega_i(x) \neq 0 \), then \( \omega_i^x \) is surjective. Also, \( \omega_i^x \) is a scalar multiple of \( \omega_i^x \) because \( \omega_i \wedge \omega_j = 0 \). Therefore, we conclude that the distribution \( \mathcal{F} \) on \( V \) is of real codimension two.

Since \( dw_i = 0 \) for all \( i \), it follows that the distribution \( \mathcal{F} \) is integrable. \( \square \)

Remark 2.2 Since \( Z(\omega_i) \) are smooth embedded submanifolds by hypothesis, and \( \omega_i \) are of type \((1,0)\), it follows that \( Z(\omega_i) \) are almost complex submanifolds of \( M \), meaning \( J_M \) preserves the tangent subbundle \( TZ(\omega_i) \subset (TM)|_{Z(\omega_i)} \). Consequently, all the intersections of the \( Z(\omega_i) \)'s are also almost complex submanifolds and are therefore even dimensional.

Clearly, each \( Z(\omega_i) \) has complex codimension at least one (real codimension at least two) as an almost complex submanifold. Therefore, the complement \( M \setminus V \) has complex codimension at least two.

2.2. Almost complex blow-up

We shall need an appropriate notion of blow-up in our context. Suppose \( K \subset M \) is a smooth embedded submanifold of \( M \) of dimension \( 2j \) such that \( J_M(TK) = TK \). By Remark 2.2, the submanifold \( \bigcap_{i=1}^g Z(\omega_i) \) satisfies these conditions. Note that \( J_M \) induces an automorphism

\[
J_{M/K} : ((TM)|_K)/TK \longrightarrow ((TM)|_K)/TK
\]

of the quotient bundle over \( K \). Since \( J_M \) is an almost complex structure it follows that \( J_{M/K} \circ J_{M/K} = -\text{Id}_{((TM)|_K)/TK} \). Therefore, \( ((TM)|_K)/TK \) is a complex vector bundle on \( K \) of rank \( k - j \). We would like to replace \( K \) by the (complex) projectivized normal bundle \( \mathbb{P}(((TM)|_K)/TK) \) which will be called the \textbf{almost complex blow-up} of \( M \) along \( K \).

For notational convenience, the intersection \( \bigcap_{i=1}^g Z(\omega_i) \) will be denoted by \( Z \).

We first projectivize \( (TM)|_Z \) to get a \( \mathbb{C}P^{k-1} \) bundle over \( Z \); this \( \mathbb{C}P^{k-1} \) bundle will be denoted by \( \mathcal{B} \). So \( \mathcal{B} \) parametrizes the space of all (real) two dimensional subspaces of \( (TM)|_Z \) preserved by \( J_M \) (such two dimensional subspaces are precisely the complex lines in \( (TM)|_Z \) equipped with the complex vector bundle structure defined by \( J_M \)). Let

\[
\pi : \mathcal{B} \longrightarrow Z
\]
be the natural projection.

For notational convenience, the pulled back vector bundle $\pi^*((TM)|_Z)$ will be denoted by $\pi^*TM$.

Let

$$T_\pi := \ker(d\pi) \subset TB$$

be the relative tangent bundle, where $d\pi : TB \longrightarrow \pi^*TZ$ is the differential of $\pi$. The map $\pi$ is almost holomorphic, and $T_\pi$ has the structure of a complex vector bundle. It is known that the complex vector bundle $T_\pi$ is identified with the vector bundle $\text{Hom}_C(\mathcal{L}, (\pi^*TM)/\mathcal{L}) = ((\pi^*TM)/\mathcal{L}) \otimes_C \mathcal{L}^*$, where

$$\mathcal{L} \subset \pi^*TM$$

is the tautological real vector bundle of rank two; note that both $\mathcal{L}$ and $(\pi^*TM)/\mathcal{L}$ have structures of complex vector bundles given by $J_M$, and the above tensor product and homomorphisms are both over $\mathbb{C}$. Therefore, the pullback $\pi^*TM$ splits as

$$\mathcal{L} \oplus ((\pi^*TM)/\mathcal{L}) = \mathcal{L} \oplus (T_\pi \otimes \mathcal{L}).$$

The image of the zero section of the complex line bundle $\mathcal{L} \longrightarrow B$ is identified with $B$, and the normal bundle of $B \subset \mathcal{L}$ is identified with $\mathcal{L}$. A small deleted normal neighborhood $U_B$ of $B$ in $\mathcal{L}$ can be identified with a deleted neighborhood $U$ of $Z$ in $M$. Let $\overline{U}_B := U_B \cup B$ be the neighborhood of $B$ in $\mathcal{L}$ (it is no longer a deleted neighborhood), where $B$ is again identified with the image of the zero section of $\mathcal{L}$. In the disjoint union $(M \setminus Z) \sqcup \overline{U}_B$, we may identify $U$ with $U_B$. The resulting topological space will be called the almost complex blow-up of $M$ along $Z$.

\textbf{Remark 2.3}

(1) Consider the foliation on the deleted neighborhood $U$ of $Z$ in $M$ defined by $\mathcal{F}$. Let $\mathbb{L}_U$ denote the leaf space for it. After identifying $B$ with the image of the zero section of $\pi^*TM$, we obtain locally, from a neighborhood of $B$, a map to the leaf space $\mathbb{L}_U$.

(2) Note that the notion of an almost complex blow-up above is well-defined up to the choice of an identification of $U$ with $U_B$. Therefore, the construction yields a well-defined almost complex manifold, up to an isomorphism.
3. Proof of Theorem 1.2

We will first show that the leaves of $F$ in Section 2 are proper embedded submanifolds of $V = M \setminus \mathcal{Z}$.

Suppose some leaf $\mathbb{L}_0$ does not satisfy the above property. Then there exists $x \in V$ and a neighborhood $U_x$ of $x$ such that $U_x \cap \mathbb{L}_0$ contains infinitely many leaves (of $F$) accumulating at $x$. Let $\sigma$ be a smooth path starting at $x$ transverse to $F$. Thus for every $\epsilon > 0$ there exist

1. distinct (local) leaves $F_1, F_2 \subset U_x \cap F$, such that (globally) $F_1, F_2 \subset \mathbb{L}_0$,
2. points $y_j \in F_j$, $j = 1, 2$,
3. a sub-path $\sigma_{12} \subset \sigma \subset U_x$ joining $y_1$ and $y_2$, and
4. a path $\tau_{12} \subset \mathbb{L}_0$ joining $y_1$ and $y_2$ such that

$$\left| \int_{\sigma_{12}} \omega_i \right| < \epsilon \ \forall \ i.$$

Since $F$ is in the kernel of each $\omega_i$,

$$\left| \int_{\tau_{12}} \omega_i \right| = 0, \ \forall \ i.$$

Setting $\epsilon$ smaller than the absolute value of any non-zero period of the $\omega_i$’s, it follows that

$$\int_{\tau_{12} \cup \sigma_{12}} \omega_i = 0 \ \ 1 \leq i \leq g.$$

Hence

$$\int_{\sigma_{12}} \omega_i = 0 \ \ \forall \ 1 \leq i \leq g.$$

Taking limits we obtain that $\omega_i(\sigma'(0)) = 0$; but this contradicts the choice that $\sigma$ is transverse to $F$. Therefore, the leaves of $F$ are proper embedded submanifolds of $V$.

Consequently, the leaf space

$$D = V/\langle F \rangle$$

for the foliation $F$ is a smooth 2-manifold. Let
be the quotient map.

**Remark 3.1**  The referee kindly pointed out to us the following considerably simpler proof of the fact that leaves of $\mathcal{F}$ are closed in $V$:

By equation (1), we have $\omega_i = f_{i,j}\omega_j$. Since $\omega_i$ are closed,

$$d(f_{i,j}) \wedge \omega_j = 0, \quad \forall \ i, j.$$  

The functions $f_{i,j}$ define a map $f : V \longrightarrow \mathbb{C}P^1$. Since $d(f_{i,j}) \wedge \omega_j = 0$, it follows that $f$ is constant on the leaves of the foliation $\mathcal{F}$ defined by the forms $\omega_i$, $1 \leq i \leq n$.

Next, let

$$\xi := TV/\mathcal{F}$$

be the quotient complex line bundle on $V$. Then $\xi$ carries a natural flat partial connection $\tilde{D}$ along the leaves (cf. [5]); this $\tilde{D}$ is known as the Bott partial connection. It is straightforward to check that $\tilde{D}$ preserves the complex structure on $\xi$. Indeed, this follows immediately from the fact that $\omega_i$ are of type $(1,0)$. Hence $\xi$ induces an almost complex structure on $D$. Since $D$ is two–dimensional, this almost complex structure is integrable giving $D$ the structure of a Riemann surface. Further, there exist closed integral $(1,0)$ forms $\eta_i$, $1 \leq i \leq g$, on $D$ such that

$$\omega_i = f^*\eta_i.$$  

**3.1. Removing indeterminacy**

Since $Z(\omega_i)$ are mutually transverse, we can construct the almost complex blow-ups of $M$ along $Z$ successively along the $\mu$–fold intersections, $\mu = 2, \ldots, g$, to obtain $\hat{M}$ such that

(1) there is an extension of the line bundle $\xi$ to all of $\hat{M}$,

(2) there is a well-defined smooth map

$$\hat{q} : \hat{M} \longrightarrow D$$

extending $q$, and
(3) the blown-up locus has the structure of complex analytic $\mathbb{CP}^1$-bundles as usual (due to transversality).

The Riemann surface $D$ is compact because $\widehat{M}$ is so. Further, $D$ has genus greater than one as $g > 1$. So any complex analytic map from $\mathbb{CP}^1$ to $D$, and we may assume that the indeterminacy locus of $q$ was empty to start off with, or equivalently that $q$ extends to a smooth map $\overline{q} : M \longrightarrow D$. This furnishes the required conclusion and completes the proof of Theorem 1.2.

4. Refinements and consequences

4.1. Stein factorization

We now proceed as in the proof of the classical Castelnuovo-de Franchis Theorem, [1, p. 24, Theorem 2.7], to deduce a posteriori that $D$ is a Stein factorization of a map to a compact Riemann surface. Define

$$h : V \longrightarrow \mathbb{CP}^{g-1}, \quad m \mapsto [\omega_1(m) : \cdots : \omega_g(m)].$$

Suppose $\omega_i(m) \neq 0$. Then there exists a small neighborhood $U(m)$ such that

$$h(x) = [f_{1,i}(x) : \cdots : f_{i-1,i}(x) : 1 : f_{i+1,i}(x) : \cdots : f_{g,i}(x)], \quad \forall \ x \in U(m),$$

where $f_{i,j}$ are defined in equation (1). Since $\omega_j \wedge \omega_i = 0$ and $d\omega_j = 0$ for all $i, j$, it follows that $df_{j,i} \wedge \omega_i = 0$. Consequently, each $f_{j,i}$ is constant on the leaves of $\mathcal{F}$. Hence $h$ has a (complex) one dimensional image, so the image of $h$ is a Riemann surface $C$.

Further, $h$ induces a holomorphic map

$$h_1 : D \longrightarrow C.$$

We note that $D$ may be thought of as the Stein factorization of $h$, and $h$ factors as $h = h_1 \circ q$.

4.2. Further generalizations

Let $M$ be a compact manifold, and let $\mathcal{S} \subset TM$ be a nonsingular foliation such that $M$ has a flat transversely almost complex structure. This means that we have an automorphism
\[ J : TM/S \rightarrow TM/S \]
such that
(1) \( J \circ J = -\text{Id}_{TM/S} \), and
(2) \( J \) is flat with respect to the Bott partial connection on \( TM/S \).

Let \( \omega_i, 1 \leq i \leq g \), be linearly independent smooth sections of \((TM/S)^* \otimes \mathbb{C}\) such that
(1) each \( \omega_i \) is flat with respect to the connection on \((TM/S)^* \otimes \mathbb{C}\) induced by the Bott partial connection on \( TM/S \),
(2) each \( \omega_i \) is of type \((1,0)\), meaning the corresponding homomorphism \( TM/S \rightarrow M \times \mathbb{C} \) is \(\mathbb{C}\)-linear,
(3) each \( \omega_i \) is closed when considered as a complex 1-form on \( M \) using the composition
\[
TM \otimes \mathbb{C} \rightarrow (TM/S) \otimes \mathbb{C} \xrightarrow{\omega_i} M \times \mathbb{C}
\]
(4) \( \omega_i \wedge \omega_j = 0 \) for all \( i, j \), and
(5) \( \omega_j \) are in general position.

Theorem 1.2 can be generalized to this set-up.

**Remark 4.1** The only real use we have made of the hypothesis in Theorem 1.2 that \( \omega_i \)’s are in general position is to ensure that \( Z \) has complex codimension at least two. This is what allows us to remove indeterminacies in the smooth category (as opposed to the algebraic or complex analytic categories, where complex codimension greater than one follows naturally). Any hypothesis that ensures that the indeterminacy locus has complex codimension greater than one would suffice.

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