Hokkaido Mathematical Journal Vol. 44 (2015) p. 441–458

Transformations between Singer-Thorpe bases in 4-dimensional Einstein manifolds

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(Received July 3, 2013; Revised September 30, 2013)

Abstract. It is well known that, at each point of a 4-dimensional Einstein Riemannian manifold (M, g), the tangent space admits at least one so-called Singer-Thorpe basis with respect to the curvature tensor R at p. K. Sekigawa put the question "how many" Singer-Thorpe bases exist for a fixed curvature tensor R. Here we work only with algebraic structures $(\mathbb{V}, \langle , \rangle, R)$, where $\langle , \rangle$ is a positive scalar product and R is an algebraic curvature tensor (in the sense of P. Gilkey) which satisfies the Einstein property. We give a partial answer to the Sekigawa problem and we state a reasonable conjecture for the general case. Moreover, we solve completely a modified problem: how many there are orthonormal bases which are Singer-Thorpe bases simultaneously for a natural 5-dimensional family of Einstein curvature tensors R. The answer is given by what we call "the universal Singer-Thorpe group" and we show that it is a finite group with 2304 elements.

Key words: Einstein manifold, 2-stein manifold, Singer-Thorpe basis.

1. Introduction

Singer and Thorpe, see [8], have proved the following:

Theorem 1 *If (M, g) is a 4-dimensional Einstein Riemannian manifold and R its curvature tensor at some fixed point p, then there is an orthonormal basis $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ in T_pM such that the complementary sectional curvatures are equal, i.e. $K_{12} = K_{34}$, $K_{13} = K_{24}$, $K_{14} = K_{23}$, and all corresponding components R_{ijkl} with exactly three distinct indices are equal to zero.*

Such a basis is referred to, standardly, as a Singer-Thorpe basis or, shortly, as an S-T basis. In the following, we shall study S-T bases in a purely algebraic way. Following P. Gilkey (see [4, p. 17]), we introduce the following

Definition 2 An *algebraic curvature tensor* on a vector space $\mathbb{V}$ with a

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2010 Mathematics Subject Classification : 53C25.

positive scalar product $\langle , \rangle$ is a tensor R of the type $(0,4)$ on $\mathbb{V}$ which satisfies the same symmetries and antisymmetries as the Riemannian curvature tensor of a Riemannian manifold, i.e.

$$\begin{aligned} R(U,V,W,Z) &= -R(V,U,W,Z) = R(W,Z,U,V), \\ R(U,V,W,Z) &+ R(V,W,U,Z) + R(W,U,V,Z) = 0 \end{aligned} \tag{1}$$

for all $U,V,W,Z \in \mathbb{V}$. Further, a triplet $(\mathbb{V}, \langle , \rangle, R)$ as above (or, an algebraic curvature tensor R on $\mathbb{V}$) is said to be Einstein if the corresponding Ricci tensor ρ on $\mathbb{V}$ satisfies the identity $\rho = \lambda \langle , \rangle$ for some $\lambda \in \mathbb{R}$.

Now, analogously as in [8], one can prove the following algebraic version of Theorem 1:

Theorem 3 *Let $\mathbb{V}$ be a 4-dimensional vector space provided with a positive scalar product $\langle , \rangle$. Let R be an Einstein algebraic curvature tensor on $\mathbb{V}$. Then there is an orthonormal basis $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ of $\mathbb{V}$ such that the nontrivial components of R with respect to $\mathcal{B}$ are, up to standard symmetries and antisymmetries, the following*:

$$\begin{aligned} &R_{1212} = R_{3434} = A, \quad &&R_{1313} = R_{2424} = B, \quad &&R_{1414} = R_{2323} = C, \\ &R_{1234} = F, &&R_{1423} = G, &&R_{1324} = F + G, \end{aligned} \tag{2}$$

where A, B, C, F, G are some constants. On the other hand, all components R_{ijkl} with exactly three distinct indices are equal to zero.

Definition 4 An orthonormal basis $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ of $\mathbb{V}$ with the properties given above is called an S-T basis on $\mathbb{V}$ corresponding to the curvature tensor R.

Definition 5 Let $(\mathbb{V}, \langle , \rangle, R)$ be an Einstein triplet. Then $\mathbb{V}$ is called 2-stein if

$$\mathcal{F}(X) = \sum_{i,j=1}^{n} (R(X, e_i, X, e_j))^2 \tag{3}$$

is independent on the choice of the unit vector $X \in \mathbb{V}$, where $\mathcal{B} = \{e_1, \ldots, e_n\}$ is any orthonormal basis. (Cf. [1].)

Then, we have the following (cf. Lemma 7 in [7]):

Proposition 6 *An Einstein triplet* $(\mathbb{V}, \langle,\rangle, R)$ *of dimension* 4 *is* 2*-stein if and only if*

$$\pm F = A - \tau/12, \qquad \mp(F+G) = B - \tau/12, \qquad \pm G = C - \tau/12 \qquad (4)$$

hold with respect to any S-T basis of $\mathbb{V}$*. Here* $\tau = \sum_{i=1}^{n} \rho(e_i, e_i)$*.*

Now, let $\{J_1, J_2, J_3\}$ be a quaternionic structure on $(\mathbb{V}, \langle,\rangle)$ compatible with a fixed orientation defined by

$$\begin{aligned} J_1X &= -x_2e_1 + x_1e_2 - x_4e_3 + x_3e_4, \\ J_2X &= -x_3e_1 + x_4e_2 + x_1e_3 - x_2e_4, \\ J_3X &= -x_4e_1 - x_3e_2 + x_2e_3 + x_1e_4 \end{aligned} \qquad (5)$$

for any $X = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 \in \mathbb{V}$, where $\{e_1, \ldots, e_4\}$ is an S-T basis compatible with the given orientation. Then, the following fact is also well known ([5], [6]):

Proposition 7 *Let* $(\mathbb{V}, \langle,\rangle, R)$ *be an Einstein triplet. Then the following two assertions are equivalent*:

(i) *For any quaternionic structure on* $\mathbb{V}$ *given by* (5) *and any unit vector* $X \in \mathbb{V}$*, the quadruplet* $\{X, J_1X, J_2X, J_3X\}$ *is an S-T basis for* R*.*

(ii) $(\mathbb{V}, \langle,\rangle, R)$ *is* 2*-stein.*

Motivated by this result and also by research in so-called weakly Einstein spaces (see [2], [3]), K. Sekigawa put the following, more general question: Let (M, g) be a 4-dimensional Einstein manifold, not necessarily 2-stein, and $\{e_1, \ldots, e_4\}$ be an arbitrary fixed S-T basis at any point $p \in M$. Determine the relation between all S-T bases $\{\bar{e}_1, \ldots, \bar{e}_4\}$ at p and the fixed S-T basis $\{e_1, \ldots, e_4\}$.

2. Algebraic preliminaries

We first notice that the relation between two S-T bases is characterized by an orthogonal transformation (i.e., by an orthogonal matrix). Let $P = (p^i_j) \in \mathrm{O}(4)$ be the matrix of an orthogonal transformation acting on the set of orthonormal bases of $(\mathbb{V}, \langle,\rangle)$ in the natural way. Hence, if $\mathcal{B} = \{e_i\}_{i=1}^4$

is an orthonormal basis, the new orthonormal basis $\mathcal{B}P = \mathcal{B}' = \{e'_j\}_{j=1}^4$ is given as $e'_j = \sum_{i=1}^4 e_i p^i_j$. Let us denote by P^{ij}_{kl} the 2×2 submatrix of the matrix P formed by the elements in the rows i, j and in the columns k, l. Let us denote by d^{ij}_{kl} its determinant. For each pair ij of indices, by $\overline{ij}$ we mean the complementary pair of indices from the set $\{1, 2, 3, 4\}$.

Lemma 8 *Any matrix $P \in \mathrm{O}(4)$ satisfies*

$$d^{ij}_{kl} \cdot d^{\overline{ij}}_{kl} = d^{ij}_{\overline{kl}} \cdot d^{\overline{ij}}_{\overline{kl}},$$
$$\left(d^{ij}_{kl}\right)^2 = \left(d^{\overline{ij}}_{\overline{kl}}\right)^2 \tag{6}$$

for arbitrary pairs ij and kl of indices from the set $\{1, 2, 3, 4\}$.

Proof. We denote by P^T the transpose of the matrix P and by E the unit matrix corresponding to the identity transformation of $\mathbb{V}$. We write down the condition $PP^T = E$ in the block form and we obtain

$$\begin{pmatrix} P^{12}_{12}(P^{12}_{12})^T + P^{12}_{34}(P^{12}_{34})^T, & P^{12}_{12}(P^{34}_{12})^T + P^{12}_{34}(P^{34}_{34})^T \\ P^{34}_{12}(P^{12}_{12})^T + P^{34}_{34}(P^{12}_{34})^T, & P^{34}_{12}(P^{34}_{12})^T + P^{34}_{34}(P^{34}_{34})^T \end{pmatrix} = \begin{pmatrix} E & N \\ N & E \end{pmatrix},$$

where N is the zero 2×2 matrix. Now, from the condition

$$P^{12}_{12}(P^{34}_{12})^T = -P^{12}_{34}(P^{34}_{34})^T,$$

we obtain

$$d^{12}_{12} \cdot d^{34}_{12} = d^{12}_{34} \cdot d^{34}_{34}.$$

In the similar way, we write down the condition $P^T P = E$. We obtain in particular

$$d^{12}_{12} \cdot d^{12}_{34} = d^{34}_{12} \cdot d^{34}_{34}.$$

The last two equations together give us

$$(d^{12}_{12})^2 = (d^{34}_{34})^2, \qquad (d^{12}_{34})^2 = (d^{34}_{12})^2.$$

The formula for arbitrary pairs of indices ij and kl can be obtained easily by the permutation of lines or columns in the matrix P. $\square$

Lemma 9 *Let $\mathcal{B}$ be an S-T basis for an Einstein algebraic curvature tensor R in which the components of R are given by (2). Then the components of the tensor R in the basis $\mathcal{B}' = \mathcal{B}P$ are given by the formula*

$$\begin{aligned} R'_{ijkl} = {} & (d^{12}_{ij} \cdot d^{12}_{kl} + d^{34}_{ij} \cdot d^{34}_{kl})\, A + (d^{13}_{ij} \cdot d^{13}_{kl} + d^{24}_{ij} \cdot d^{24}_{kl})\, B \\ & + (d^{14}_{ij} \cdot d^{14}_{kl} + d^{23}_{ij} \cdot d^{23}_{kl})\, C \\ & + (d^{12}_{ij} \cdot d^{34}_{kl} + d^{34}_{ij} \cdot d^{12}_{kl} + d^{13}_{ij} \cdot d^{24}_{kl} + d^{24}_{ij} \cdot d^{13}_{kl})\, F \\ & + (d^{14}_{ij} \cdot d^{23}_{kl} + d^{23}_{ij} \cdot d^{14}_{kl} + d^{13}_{ij} \cdot d^{24}_{kl} + d^{24}_{ij} \cdot d^{13}_{kl})\, G. \end{aligned} \tag{7}$$

Proof. It follows by the straightforward check using formulas

$$R'_{ijkl} = R(e'_i, e'_j, e'_k, e'_l),$$

where the components of the vector e'_i are p^u_i (the i-th column of the given matrix P). $\square$

Corollary 10 *Let $\mathcal{B}$ be an S-T basis for an Einstein algebraic curvature tensor R. For any matrix $P \in \mathrm{O}(4)$, the components of the tensor R in the basis $\mathcal{B}' = \mathcal{B}P$ satisfy*

$$R'_{1212} = R'_{3434}, \quad R'_{1313} = R'_{2424}, \quad R'_{1414} = R'_{2323}.$$

Proof. From Lemma 9 and formula (6) written for the matrix P^T, we obtain

$$\begin{aligned} R'_{1212} = R'_{3434} = {} & \big((d^{12}_{12})^2 + (d^{34}_{12})^2\big)\, A + \big((d^{13}_{12})^2 + (d^{24}_{12})^2\big)\, B \\ & + \big((d^{14}_{12})^2 + (d^{23}_{12})^2\big)\, C + 2\big(d^{12}_{12} \cdot d^{34}_{12} + d^{13}_{12} \cdot d^{24}_{12}\big)\, F \\ & + 2\big(d^{14}_{12} \cdot d^{23}_{12} + d^{13}_{12} \cdot d^{24}_{12}\big)\, G, \\ R'_{1313} = R'_{2424} = {} & \big((d^{12}_{13})^2 + (d^{34}_{13})^2\big)\, A + \big((d^{13}_{13})^2 + (d^{24}_{13})^2\big)\, B \\ & + \big((d^{14}_{13})^2 + (d^{23}_{13})^2\big)\, C + 2\big(d^{12}_{13} \cdot d^{34}_{13} + d^{13}_{13} \cdot d^{24}_{13}\big)\, F \\ & + 2\big(d^{14}_{13} \cdot d^{23}_{13} + d^{13}_{13} \cdot d^{24}_{13}\big)\, G, \end{aligned}$$

$$\begin{aligned} R'_{1414} = R'_{2323} = {} & \big((d^{12}_{14})^2 + (d^{34}_{14})^2\big)\, A + \big((d^{13}_{14})^2 + (d^{24}_{14})^2\big)\, B \\ & + \big((d^{14}_{14})^2 + (d^{23}_{14})^2\big)\, C + 2\big(d^{12}_{14} \cdot d^{34}_{14} + d^{13}_{14} \cdot d^{24}_{14}\big)\, F \\ & + 2\big(d^{14}_{14} \cdot d^{23}_{14} + d^{13}_{14} \cdot d^{24}_{14}\big)\, G. \qquad \square \end{aligned}$$

3. The basic finite group of transformations

Let $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ be an S-T basis for an Einstein algebraic curvature tensor R on $(\mathbb{V}, \langle , \rangle)$. We are interested in transformations $P \in \mathrm{O}(4)$ such that the components of the tensor R in the new bases $\mathcal{B}' = \mathcal{B}P$ have all components with just three different indices equal to zero, namely $R'_{ijkl} = 0$ for the following 12 choices of i, j, k, l:

$$\begin{aligned} & 1213, \\ & 1214, 1314, \\ & 1223, 1323, 1424, 2324, \\ & 1224, 1334, 1434, 2334, 2434. \end{aligned} \tag{8}$$

Equivalently, all the bases $\mathcal{B}' = \mathcal{B}P$ should be new S-T bases for the tensor R. Let us denote by $\mathcal{H}_1 \subset \mathrm{O}(4)$ the group of all permutation matrices (i.e., the orthogonal matrices corresponding to permutations of the vectors e_1, e_2, e_3, e_4) and by $\mathcal{H}_2 \subset \mathrm{O}(4)$ the group of all diagonal matrices with ± 1 on the diagonal. We will denote by $|\mathcal{H}|$ the number of elements of a group $\mathcal{H}$. Obviously, $|\mathcal{H}_1| = 24$ and $|\mathcal{H}_2| = 16$. We further denote $\mathcal{H}_3 = \mathcal{H}_1 \cdot \mathcal{H}_2 = \mathcal{H}_2 \cdot \mathcal{H}_1$. We easily see that $\mathcal{H}_3$ is a group and $|\mathcal{H}_3| = 16 \cdot 24 = 384$. It is not hard to verify that, for all $P \in \mathcal{H}_3$, $\mathcal{B}P$ are S-T bases for R.

Let us consider the two special transformations given by the matrices

$$P_4 = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \qquad P_5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

The direct calculation using formula (7) shows that the components of the tensor R in the basis $\mathcal{B}' = \mathcal{B}P_4$ are

$$
\begin{aligned}
A' &= R'_{1212} = R'_{3434} = 1/2(B + C - F - 2G),\\
B' &= R'_{1313} = R'_{2424} = 1/2(A + C - F + G),\\
C' &= R'_{1414} = R'_{2323} = 1/2(A + B + 2F + G),\\
F' &= R'_{1234} \qquad\qquad\; = 1/2(-B + C + F),\\
G' &= R'_{1423} \qquad\qquad\; = 1/2(-A + B + G).
\end{aligned} \tag{9}
$$

and $R'_{ijkl} = 0$ for all $ijkl$ from (8). In the basis $\mathcal{B}' = \mathcal{B}P_5$, the components of the tensor R are

$$
\begin{aligned}
A' &= R'_{1212} = R'_{3434} = A,\\
B' &= R'_{1313} = R'_{2424} = 1/2(B + C + F + 2G),\\
C' &= R'_{1414} = R'_{2323} = 1/2(B + C - F - 2G),\\
F' &= R'_{1234} \qquad\qquad\; = F,\\
G' &= R'_{1423} \qquad\qquad\; = 1/2(B - C - F)
\end{aligned} \tag{10}
$$

and $R'_{ijkl} = 0$ for all $ijkl$ from (8). We see that both $\mathcal{B}P_4$ and $\mathcal{B}P_5$ are S-T bases for R. Let us denote by P'_4 the transformation which has the first three columns same as the transformation P_4 and the last column with the opposite sign. Obviously, $P'_4 \in P_4\mathcal{H}_2$ and $\mathcal{B}P'_4$ is also an S-T basis for R.

Lemma 11 *The group $\mathcal{H}_4$ generated by $\mathcal{H}_3$ and P_4 is the union of cosets $\mathcal{H}_3 \cup \mathcal{H}_3P_4 \cup \mathcal{H}_3P'_4$.*

Proof. Let $x \in P_4\mathcal{H}_3$. We can write $x = P_4ps$, where $p \in \mathcal{H}_1$ and $s \in \mathcal{H}_2$. There is an element $p' \in \mathcal{H}_1$ such that $p'P_4 = P_4p$ and hence $x \in \mathcal{H}_3P_4s$. If s has an even number of entries equal to -1, then there is an element $y \in \mathcal{H}_3$ such that $P_4s = yP_4$. If s has on odd number of entries equal to -1, then there is an element $y \in \mathcal{H}_3$ such that $P_4s = yP'_4$. Hence either $x \in \mathcal{H}_3P_4$ or $x \in \mathcal{H}_3P'_4$. Let $x \in P_4\mathcal{H}_3P_4$. Either $x \in \mathcal{H}_3P_4P_4 = \mathcal{H}_3E = \mathcal{H}_3$ or $x \in \mathcal{H}_3P'_4P_4 = \mathcal{H}_3P'_4$, because $P'_4P_4 = sP'_4$ for $s \in \mathcal{H}_2$. It finishes the proof. □

Corollary 12 *It holds $|\mathcal{H}_4| = 3 \cdot 384 = 1152$.*

Lemma 13 *The group $\mathcal{H}_5$ generated by $\mathcal{H}_4$ and P_5 is the union of cosets $\mathcal{H}_4 \cup \mathcal{H}_4P_5$.*

Proof. By the direct calculations we obtain $|\mathcal{H}_3 P_5 \mathcal{H}_3| = 3 \cdot 6 \cdot 8 \cdot 8 = 1152$. It is not hard to verify that $\mathcal{H}_3 P_5 \mathcal{H}_3 \subset \mathcal{H}_4 P_5$ and $\mathcal{H}_3 P_5 \mathcal{H}_3 \subset P_5 \mathcal{H}_4$, hence we obtain $\mathcal{H}_4 P_5 = \mathcal{H}_3 P_5 \mathcal{H}_3 = P_5 \mathcal{H}_4$. Obviously $\mathcal{H}_5 = \mathcal{H}_4 \cup \mathcal{H}_4 P_5$. □

Corollary 14 *It holds* $|\mathcal{H}_5| = 2 \cdot 1152 = 2304$.

Because the group $\mathcal{H}_5$ is generated by the subgroup $\mathcal{H}_3$ and the matrices P_4, P_4', P_5, we see that the conditions $R'_{ijkl} = 0$ for the 12 choices of indices $ijkl$ as in (8) hold for any $P \in \mathcal{H}_5$. In other words, for all $P \in \mathcal{H}_5$, the bases $\mathcal{B}P$ are S-T bases for R.

4. The universal Singer-Thorpe group

In the following, for a given matrix $M \in \mathrm{O}(4)$, all matrices from the set $\mathcal{H}_3 M \mathcal{H}_3$ will be called *matrices of type* M. In other words, a matrix M' of type M arises by a permutation of the rows, a permutation of the columns and by changing the sign of any row or column in the matrix M.

Let us now fix an orthonormal basis $\mathcal{B}$ of $(\mathbb{V}, \langle , \rangle)$ and consider the set of *all* algebraic curvature tensors for which $\mathcal{B}$ is an S-T basis. (These curvature tensors depend on 5 parameters A, B, C, F, G and they are automatically Einstein.) Let us denote by $\mathcal{S}$ the set of bases which are S-T bases *for all these tensors*. Finally, let us denote by $\mathcal{G}$ the set of orthogonal matrices corresponding to all transformations between the bases from $\mathcal{S}$ when expressed with respect to the basis $\mathcal{B}$. Then, obviously, $\mathcal{G} \subset \mathrm{O}(4)$ is a group and it is independent of the initial basis $\mathcal{B}$.

Theorem 15 *The group* $\mathcal{G}$ *is just the group* $\mathcal{H}_5$ *of* 2304 *elements described in Section* 3.

Proof. We denote the coefficients in the formula (7) by A'_{ijkl}, B'_{ijkl}, C'_{ijkl}, F'_{ijkl}, G'_{ijkl} and write the formula in the short form

$$R'_{ijkl} = A'_{ijkl}A + B'_{ijkl}B + C'_{ijkl}C + F'_{ijkl}F + G'_{ijkl}G. \tag{11}$$

Because A, B, C, F, G can be considered as independent variables, it is clear that for $P \in \mathcal{G}$ it must hold $A'_{ijkl} = B'_{ijkl} = C'_{ijkl} = F'_{ijkl} = G'_{ijkl} = 0$ for the 12 choices of indices from (8). According to the end of Section 3, we have the inclusion $\mathcal{H}_5 \subset \mathcal{G}$. We only have to prove that $\mathcal{G} \subset \mathcal{H}_5$. This will be done, step by step, in the rest of this Section.

Lemma 16 *Let the transformation P satisfy $A'_{ijkl} = B'_{ijkl} = C'_{ijkl} = 0$ for all 12 choices of $ijkl$ from (8). Then there exist three pairs of indices, not necessarily distinct, such that*

$$d^{12}_{pr} = d^{12}_{\overline{pr}} = d^{34}_{pr} = d^{34}_{\overline{pr}} = 0,$$
$$d^{13}_{st} = d^{13}_{\overline{st}} = d^{24}_{st} = d^{24}_{\overline{st}} = 0,$$
$$d^{14}_{uv} = d^{14}_{\overline{uv}} = d^{23}_{uv} = d^{23}_{\overline{uv}} = 0. \tag{12}$$

Proof. Let $A'_{ijkl} = 0$ for all 12 choices of $ijkl$ from (8). Let us rearrange the columns of the matrix M corresponding to the transformation P in a way that $d^{12}_{12} \neq 0$. We rewrite the equalities

$$A'_{ijkl} = d^{12}_{ij} \cdot d^{12}_{kl} + d^{34}_{ij} \cdot d^{34}_{kl} = 0 \tag{13}$$

for the four choices of $ijkl = 1213, 1214, 1223, 1224$ in the forms

$$d^{12}_{13} = -d^{34}_{12} \cdot d^{34}_{13}/d^{12}_{12},$$
$$d^{12}_{14} = -d^{34}_{12} \cdot d^{34}_{14}/d^{12}_{12},$$
$$d^{12}_{23} = -d^{34}_{12} \cdot d^{34}_{23}/d^{12}_{12},$$
$$d^{12}_{24} = -d^{34}_{12} \cdot d^{34}_{24}/d^{12}_{12}.$$

We now substitute these formulas into formulas (13) for $ijkl$ equal to $1314, 1323$, $1424, 2324$, which are

$$d^{12}_{13} \cdot d^{12}_{14} + d^{34}_{13} \cdot d^{34}_{14} = 0,$$
$$d^{12}_{13} \cdot d^{12}_{23} + d^{34}_{13} \cdot d^{34}_{23} = 0,$$
$$d^{12}_{14} \cdot d^{12}_{24} + d^{34}_{14} \cdot d^{34}_{24} = 0,$$
$$d^{12}_{23} \cdot d^{12}_{24} + d^{34}_{23} \cdot d^{34}_{24} = 0.$$

From here we obtain, after elementary operations with fractions, the formulas

$$\left[(d^{34}_{12})^2 + (d^{12}_{12})^2\right] \cdot d^{34}_{13} \cdot d^{34}_{14} = 0,$$
$$\left[(d^{34}_{12})^2 + (d^{12}_{12})^2\right] \cdot d^{34}_{13} \cdot d^{34}_{23} = 0,$$

$$\left[(d_{12}^{34})^2 + (d_{12}^{12})^2\right] \cdot d_{14}^{34} \cdot d_{24}^{34} = 0,$$

$$\left[(d_{12}^{34})^2 + (d_{12}^{12})^2\right] \cdot d_{23}^{34} \cdot d_{24}^{34} = 0.$$

To satisfy these conditions, it must hold $d_{13}^{34} = d_{24}^{34} = 0$ or $d_{14}^{34} = d_{23}^{34} = 0$. According to formula (6), the first of these equalities imply $d_{13}^{12} = d_{24}^{12} = 0$ and the second equalities imply $d_{14}^{12} = d_{23}^{12} = 0$, which proves the conditions in the first line of the statement. The conditions in the second and the third line of the statement can be proved analogously using conditions $B'_{ijkl} = 0$ and $C'_{ijkl} = 0$. □

Obviously, the same statement must be true for the inverse transformation P^T. We can reformulate it in the way that in any two rows, or columns, there are two complementary subdeterminants which are both zero and in the remaining two lines, or columns, the same subdeterminants are zero too. In the sequel, we shall assume that in the lines 12 and 34 these determinants belong to columns 12 and 34, hence $d_{12}^{12} = d_{34}^{12} = d_{12}^{34} = d_{34}^{34} = 0$. First, we exclude the case when pairs of indices in Lemma 16 are not distinct.

Lemma 17 *Let the matrix $P \in \mathrm{O}(4)$ satisfy $d_{12}^{12} = d_{34}^{12} = d_{12}^{34} = d_{34}^{34} = 0$. If $d_{12}^{13} = d_{34}^{13} = d_{12}^{24} = d_{34}^{24} = 0$ or $d_{12}^{14} = d_{34}^{14} = d_{12}^{23} = d_{34}^{23} = 0$, then the matrix P either belongs to $\mathcal{H}_3$ or it is of the type P_5.*

Proof. In the submatrix of P formed by columns 1 and 2, there must be two independent rows, because this submatrix has rank 2. Let these rows be the k-th and the l-th. According to the assumptions $d_{12}^{12} = d_{12}^{34} = 0$, we cannot have $kl = 12$ or $kl = 34$. Further, according to the assumptions $d_{12}^{13} = d_{12}^{24} = 0$, the remaining two rows are simultaneously multiples of the k-th row and the l-th row. Hence they are zero vectors and the submatrix $P_{12}^{\overline{kl}}$ is the zero matrix. In the same way, for the columns 3 and 4 we obtain that the submatrix P_{34}^{kl} is the zero matrix. Without the loss of generality, we will assume $kl = 23$. The matrix $P \in \mathrm{O}(4)$ is in the form

$$\begin{pmatrix} 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & \cdot \end{pmatrix}.$$

Now, we obtain easily that $C'_{ijkl} = 0$ for all 12 choices of $ijkl$ from (8) and

$A'_{ijkl} = B'_{ijkl} = 0$ for $ij = 12$ or $kl = 34$. The nonzero conditions are hence $A'_{ijkl} = 0$ and $B'_{ijkl} = 0$ for $ijkl = 1314, 1323, 1424, 2324$. Using formula (7), these conditions are

$$\begin{aligned} d^{12}_{13} \cdot d^{12}_{14} + d^{34}_{13} \cdot d^{34}_{14} &= 0, & d^{13}_{13} \cdot d^{13}_{14} + d^{24}_{13} \cdot d^{24}_{14} &= 0, \\ d^{12}_{13} \cdot d^{12}_{23} + d^{34}_{13} \cdot d^{34}_{23} &= 0, & d^{13}_{13} \cdot d^{13}_{23} + d^{24}_{13} \cdot d^{24}_{23} &= 0, \\ d^{12}_{14} \cdot d^{12}_{24} + d^{34}_{14} \cdot d^{34}_{24} &= 0, & d^{13}_{14} \cdot d^{13}_{24} + d^{24}_{14} \cdot d^{24}_{24} &= 0, \\ d^{12}_{23} \cdot d^{12}_{24} + d^{34}_{23} \cdot d^{34}_{24} &= 0, & d^{13}_{23} \cdot d^{13}_{24} + d^{24}_{23} \cdot d^{24}_{24} &= 0. \end{aligned}$$

Writing down the determinants d^{ij}_{kl} explicitly, we obtain

$$\begin{aligned} (p^2_1)^2 \cdot p^1_3 \cdot p^1_4 + (p^3_1)^2 \cdot p^4_3 \cdot p^4_4 &= 0, & (p^3_1)^2 \cdot p^1_3 \cdot p^1_4 + (p^2_1)^2 \cdot p^4_3 \cdot p^4_4 &= 0, \\ (p^2_2)^2 \cdot p^1_3 \cdot p^1_4 + (p^3_2)^2 \cdot p^4_3 \cdot p^4_4 &= 0, & (p^3_2)^2 \cdot p^1_3 \cdot p^1_4 + (p^2_2)^2 \cdot p^4_3 \cdot p^4_4 &= 0, \\ p^2_1 \cdot p^2_2 \cdot (p^1_3)^2 + p^3_1 \cdot p^3_2 \cdot (p^4_3)^2 &= 0, & p^3_1 \cdot p^3_2 \cdot (p^1_3)^2 + p^2_1 \cdot p^2_2 \cdot (p^4_3)^2 &= 0, \\ p^2_1 \cdot p^2_2 \cdot (p^1_4)^2 + p^3_1 \cdot p^3_2 \cdot (p^4_4)^2 &= 0, & p^3_1 \cdot p^3_2 \cdot (p^1_4)^2 + p^2_1 \cdot p^2_2 \cdot (p^4_4)^2 &= 0. \end{aligned} \tag{14}$$

In the case $p^i_j = 0$ for some p^i_j in formulas (14), we obtain from these conditions and the condition $P \in \mathrm{O}(4)$ that $P \in \mathcal{H}_3$. In the case $p^i_j \neq 0$ for all p^i_j which appear in formulas (14), we obtain by the elementary operations the conditions

$$\begin{aligned} (p^1_3)^2 \cdot (p^1_4)^2 &= (p^4_3)^2 \cdot (p^4_4)^2, & (p^1_3)^4 &= (p^4_3)^4, & (p^1_4)^4 &= (p^4_4)^4, \\ (p^2_1)^2 \cdot (p^2_2)^2 &= (p^3_1)^2 \cdot (p^3_2)^2, & (p^2_1)^4 &= (p^3_1)^4, & (p^2_2)^4 &= (p^3_2)^4. \end{aligned}$$

From here and from the condition $P \in \mathrm{O}(4)$ it follows that P is of the type P_5. $\square$

We are left with the case when the pairs pr, st, uv of indices in Lemma 16 are distinct. Without the loss of generality we can assume $pr = 12$, $st = 13$, $uv = 14$.

Lemma 18 *Let the matrix $P \in \mathrm{O}(4)$ satisfy formulas* (12) *with $pr = 12$, $st = 13$, $uv = 14$. Then it must be of the type P_4.*

Proof. First, we show that $p^i_j \neq 0$ for all $i, j = 1, \ldots, 4$. Let us suppose the opposite, hence $p_1^1 = 0$. From the condition $d_{12}^{12} = 0$ we obtain either $p_2^1 = 0$ or $p_1^2 = 0$. From the condition $d_{13}^{13} = 0$ we obtain either $p_3^1 = 0$ or $p_1^3 = 0$. From the condition $d_{14}^{14} = 0$ we obtain either $p_4^1 = 0$ or $p_1^4 = 0$. Using all 12 conditions (12), we obtain a contradiction with $P \in \mathrm{O}(4)$.

Hence it holds $p_j^i \neq 0$ for all $i, j = 1, \ldots, 4$. We can denote the entries in the first row and the first column as $p_1^1 = a$, $p_2^1 = ka$, $p_3^1 = ra$, $p_4^1 = ua$ and $p_2^1 = xa$, $p_3^2 = ya$, $p_4^2 = za$ for nonzero $a, x, y, z, k, r, u \in \mathbb{R}$. From the conditions $d_{12}^{12} = d_{13}^{13} = d_{14}^{14} = 0$ we obtain the entries p_2^2, p_3^3, p_4^4 and the matrix P must be in the form

$$\begin{pmatrix} a & ka & ra & ua \\ xa & xka & \cdot & \cdot \\ ya & \cdot & yra & \cdot \\ za & \cdot & \cdot & zua \end{pmatrix}.$$

Further, from the conditions $d_{12}^{34} = d_{13}^{24} = d_{14}^{23} = 0$ we obtain the other entries and the matrix P is in the form

$$\begin{pmatrix} a & ka & ra & ua \\ xa & xka & xsa & xva \\ ya & yla & yra & yva \\ za & zla & zsa & zua \end{pmatrix}$$

for some nonzero $l, s, v \in \mathbb{R}$. Now, the conditions $d_{34}^{12} = d_{24}^{13} = d_{23}^{14} = 0$ and $d_{34}^{34} = d_{24}^{24} = d_{23}^{23} = 0$ imply

$$\begin{aligned} rv &= su, \quad kv = lu, \quad ks = lr, \\ ru &= sv, \quad ku = lv, \quad kr = ls. \end{aligned}$$

From here it follows $r^2 = s^2$, $k^2 = l^2$, $u^2 = v^2$. Each of the possibilities $k = l$, $r = s$ and $u = v$ leads to a contradiction with $P \in \mathrm{O}(4)$, so it holds $k = -l$, $r = -s$ and $u = -v$. From the conditions $\sum_s p_s^i p_s^j = 0$ for $i \neq j$ we obtain

$$\begin{aligned} 1 + k^2 - r^2 - u^2 &= 0, \\ 1 - k^2 + r^2 - u^2 &= 0, \\ 1 - k^2 - r^2 + u^2 &= 0 \end{aligned}$$

which implies

$$k^2 = r^2 = u^2 = 1.$$

From the conditions $\sum_s p_i^s p_j^s = \delta_{ij}$ we obtain that we can choose $a = 1/2$ and we get $x^2 = y^2 = z^2 = 1$. The statement follows easily. $\square$

This completes the proof of Theorem 15. The group $\mathcal{G} = \mathcal{H}_5$ can be called *the universal Singer-Thorpe group.* $\square\square$

5. The set of all S-T bases for a fixed tensor R

In this Section we try to assault the original problem put by K. Sekigawa, i.e., the question what are the properties of the set of all S-T bases for a *fixed* Einstein algebraic curvature tensor R. More precisely, we shall investigate the structure of the set $\mathcal{K}(\mathcal{B}, R)$ of all orthogonal matrices corresponding to all transformations between a fixed S-T basis $\mathcal{B}$ and other S-T bases (expressed with respect to $\mathcal{B}$) for a fixed tensor R. In particular, we show that, in a special case, the set $\mathcal{K}(\mathcal{B}, R)$ may be infinite and not a group. In any case, from Section 4 we see that always $\mathcal{H}_5 \subset \mathcal{K}(\mathcal{B}, R)$. We were unable to solve the problem in general, and so, we shall conclude our study with a reasonable Conjecture. We shall start with the easiest special case, where $\mathcal{K}(\mathcal{B}, R) = \mathrm{O}(4)$.

Lemma 19 *For any matrix $P = (p^\alpha_\beta) \in \mathrm{O}(4)$ it holds*

$$\begin{aligned} &A'_{ijkl} + B'_{ijkl} + C'_{ijkl} \\ &\quad = \Big(\sum_\alpha p_i^\alpha p_k^\alpha\Big) \cdot \Big(\sum_\beta p_j^\beta p_l^\beta\Big) - \Big(\sum_\gamma p_i^\gamma p_l^\gamma\Big) \cdot \Big(\sum_\delta p_j^\delta p_k^\delta\Big) \end{aligned} \tag{15}$$

for any fixed i, j, k, l from the set $\{1, 2, 3, 4\}$.

Proof. On the left-hand side, there are coefficients from the formula (11). We use their long form from (7) and continue by the straightforward calculations:

$$\begin{aligned} &A'_{ijkl} + B'_{ijkl} + C'_{ijkl} \\ &\quad = (d^{12}_{ij} \cdot d^{12}_{kl} + d^{34}_{ij} \cdot d^{34}_{kl}) + (d^{13}_{ij} \cdot d^{13}_{kl} + d^{24}_{ij} \cdot d^{24}_{kl}) + (d^{14}_{ij} \cdot d^{14}_{kl} + d^{23}_{ij} \cdot d^{23}_{kl}) \end{aligned}$$

$$
\begin{aligned}
&= (p_i^1 p_j^2 - p_j^1 p_i^2)(p_k^1 p_l^2 - p_l^1 p_k^2) + (p_i^3 p_j^4 - p_j^3 p_i^4)(p_k^3 p_l^4 - p_l^3 p_k^4) \\
&\quad + (p_i^1 p_j^3 - p_j^1 p_i^3)(p_k^1 p_l^3 - p_l^1 p_k^3) + (p_i^2 p_j^4 - p_j^2 p_i^4)(p_k^2 p_l^4 - p_l^2 p_k^4) \\
&\quad + (p_i^1 p_j^4 - p_j^1 p_i^4)(p_k^1 p_l^4 - p_l^1 p_k^4) + (p_i^2 p_j^3 - p_j^2 p_i^3)(p_k^2 p_l^3 - p_l^2 p_k^3) \\
&= p_i^1 p_j^2 p_k^1 p_l^2 + p_j^1 p_i^2 p_l^1 p_k^2 - p_i^1 p_j^2 p_l^1 p_k^2 - p_j^1 p_i^2 p_k^1 p_l^2 + p_i^3 p_j^4 p_k^3 p_l^4 + p_j^3 p_i^4 p_l^3 p_k^4 \\
&\quad - p_i^3 p_j^4 p_l^3 p_k^4 - p_j^3 p_i^4 p_k^3 p_l^4 + p_i^1 p_j^3 p_k^1 p_l^3 + p_j^1 p_i^3 p_l^1 p_k^3 - p_i^1 p_j^3 p_l^1 p_k^3 - p_j^1 p_i^3 p_k^1 p_l^3 \\
&\quad + p_i^2 p_j^4 p_k^2 p_l^4 + p_j^2 p_i^4 p_l^2 p_k^4 - p_i^2 p_j^4 p_l^2 p_k^4 - p_j^2 p_i^4 p_k^2 p_l^4 + p_i^1 p_j^4 p_k^1 p_l^4 + p_j^1 p_i^4 p_l^1 p_k^4 \\
&\quad - p_i^1 p_j^4 p_l^1 p_k^4 - p_j^1 p_i^4 p_k^1 p_l^4 + p_i^2 p_j^3 p_k^2 p_l^3 + p_j^2 p_i^3 p_l^2 p_k^3 - p_i^2 p_j^3 p_l^2 p_k^3 - p_j^2 p_i^3 p_k^2 p_l^3 \\
&= p_i^1 p_k^1 p_j^2 p_l^2 + p_j^1 p_l^1 p_i^2 p_k^2 + p_i^1 p_k^1 p_j^3 p_l^3 + p_j^1 p_l^1 p_i^3 p_k^3 + p_i^1 p_k^1 p_j^4 p_l^4 + p_j^1 p_l^1 p_i^4 p_k^4 \\
&\quad + p_i^2 p_k^2 p_j^3 p_l^3 + p_j^2 p_l^2 p_i^3 p_k^3 + p_i^2 p_k^2 p_j^4 p_l^4 + p_j^2 p_l^2 p_i^4 p_k^4 + p_i^3 p_k^3 p_j^4 p_l^4 + p_j^3 p_l^3 p_i^4 p_k^4 \\
&\quad - p_i^1 p_l^1 p_j^2 p_k^2 - p_j^1 p_k^1 p_i^2 p_l^2 - p_i^1 p_l^1 p_j^3 p_k^3 - p_j^1 p_k^1 p_i^3 p_l^3 - p_i^1 p_l^1 p_j^4 p_k^4 - p_j^1 p_k^1 p_i^4 p_l^4 \\
&\quad - p_i^2 p_l^2 p_j^3 p_k^3 - p_j^2 p_k^2 p_i^3 p_l^3 - p_i^2 p_l^2 p_j^4 p_k^4 - p_j^2 p_k^2 p_i^4 p_l^4 - p_i^3 p_l^3 p_j^4 p_k^4 - p_j^3 p_k^3 p_i^4 p_l^4 \\
&= \Big(\sum_\alpha p_i^\alpha p_k^\alpha\Big) \cdot \Big(\sum_\beta p_j^\beta p_l^\beta\Big) - \Big(\sum_\gamma p_i^\gamma p_l^\gamma\Big) \cdot \Big(\sum_\delta p_j^\delta p_k^\delta\Big). \qquad \square
\end{aligned}
$$

Proposition 20 *Let the components of the Einstein algebraic curvature tensor R in the S-T basis $\mathcal{B}$ satisfy $A = B = C$ and $F = G = 0$. Then $\mathcal{B}' = \mathcal{B}P$ is an S-T basis for the tensor R for all transformations $P \in \mathrm{O}(4)$. In all S-T bases $\mathcal{B}' = \mathcal{B}P$, it holds $F' = G' = 0$.*

Proof. The right-hand side of the formula (15) is obviously zero whenever at least three indices among i, j, k, l are distinct. Applying our assumptions to formula (11), we get

$$R'_{ijkl} = (A'_{ijkl} + B'_{ijkl} + C'_{ijkl}) \cdot A = 0$$

for the 12 choices of $ijkl$ from the list (8) and, moreover, $F' = R'_{1234} = 0$ and $G' = R'_{1423} = 0$. $\square$

Proposition 21 *The property $A = B = C$ is not invariant under the transformation group $\mathcal{H}_5$ unless $F = G = 0$. The property $F = G = 0$ is not invariant under the group $\mathcal{H}_5$ unless $A = B = C$.*

Proof. It follows for example from formulas (9) and (10) for transformations $P_4, P_5 \in \mathcal{G}$. $\square$

We conclude this easy special case by a straightforward Corollary:

Corollary 22 *Let (M, g) be a 4-dimensional Einstein manifold. The following conditions are equivalent*:

1) *For any point $p \in M$, and for any S-T basis in T_pM with respect to the curvature tensor R_p, all components of R_p with four distinct indices are zero*;
2) *(M, g) is a space of constant curvature.*

We shall continue by the less special case, which is more interesting.

Proposition 23 *Let the components of an Einstein algebraic curvature tensor R in a given S-T basis $\mathcal{B}$ satisfy $B = C$, $F = G = 0$ and A be arbitrary. Let us consider the group of matrices*

$$M_{(s,t)} = \begin{pmatrix} \cos(t) & \sin(t) & 0 & 0 \\ -\sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & \cos(s) & \sin(s) \\ 0 & 0 & -\sin(s) & \cos(s) \end{pmatrix}, \qquad s, t \in \mathbb{R}.$$

The bases $\mathcal{B}' = \mathcal{B}M_{(s,t)}$ are S-T bases for the tensor R and components of R in all these bases satisfy $A' = A$, $B' = C' = B$, $F' = G' = 0$.

Proof. For the transformations corresponding to the matrix $M_{(s,t)}$, we have $A'_{ijkl} = d^{12}_{ij} \cdot d^{12}_{kl} + d^{34}_{ij} \cdot d^{34}_{kl} = 0$ for all 12 choices of $ijkl$ from (8). The formula (15) implies, for the same indices, $B'_{ijkl} + C'_{ijkl} = 0$ and according to formula (11) we obtain $R'_{ijkl} = 0$ for all 12 choices of $ijkl$ from (8). Hence all the new bases are S-T bases. Using formulas (11) and (15), we calculate easily $A' = A$, $B' = C' = B$, $F' = G' = 0$. $\square$

Corollary 24 *Let the components of an Einstein algebraic curvature tensor R in a given S-T basis $\mathcal{B}$ satisfy $B = C$, $F = G = 0$ and $A \neq B$ be arbitrary. Then the set $\mathcal{K}(\mathcal{B}, R)$ contains the groups $\mathcal{H}_5$ and $M_{(s,t)}$, but it is not a group in itself.*

Proof. The first assertion is obvious from the text before Lemma 19 and from Proposition 23. Let us consider the tensor R in a given S-T basis $\mathcal{B}$

whose components satisfy the assumptions. Now we apply the transformation $P_4 \in \mathcal{H}_5$ to $\mathcal{B}$. For the components of the tensor R in the basis $\mathcal{B}P_4$ we obtain, according to formula (9), $G' \neq 0$. For the new components we shall write again A, B, C, F, G instead of A', B', C', F', G'. Now we apply the transformation $M_{(s,t)}$ depending on two parameters. Then a simple generalization of Proposition 23 gives, in such a case,

$$\begin{aligned}
A' &= A,\\
B' &= 2\cos(s)\cos(t)\sin(t)\sin(s)(F+2G)+B,\\
C' &= -2\cos(s)\cos(t)\sin(t)\sin(s)(F+2G)+B,\\
F' &= F,\\
G' &= (2\cos^2(s)\cos^2(t)-\cos^2(s)-\cos^2(t))(F+2G)+G
\end{aligned}$$

and, moreover,

$$\begin{aligned}
R'_{1314} &= -\cos(t)\sin(t)\big[(4\cos^2(s)-2)G+(2\cos^2(s)-1)F\big],\\
R'_{1323} &= -\sin(s)\cos(s)\big[(4\cos^2(t)-2)G+(2\cos^2(t)-1)F\big].
\end{aligned}$$

Thus the product matrix $P = P_4M_{(s,t)}$ applied to the S-T basis $\mathcal{B}$ gives us the orthogonal basis $\mathcal{B}' = \mathcal{B}P$ which is usually not an S-T basis with respect to R, because the components of the curvature tensor R with three distinct indices in $\mathcal{B}'$ are nonzero, in general. Hence the product $P = P_4M_{(s,t)}$ falls outside the set $\mathcal{K}(\mathcal{B}, R)$, in general. On the other hand, both $M_{(s,t)}$ and P_4 belong to $\mathcal{K}(\mathcal{B}, R)$. We see that the basic group property for $\mathcal{K}(\mathcal{B}, R)$ is not satisfied. □

Proposition 25 *If the structure* $(\mathbb{V}, \langle,\rangle, R)$ *satisfies the* 2*-stein property from Proposition* 6 *and* $\mathcal{B}$ *is an S-T basis, then we have the inclusion* $\mathrm{Sp}(1) \subset \mathcal{K}(\mathcal{B}, R)$*, where* $\mathrm{Sp}(1) \subset \mathrm{O}(4)$ *is the compact symplectic group.*

Proof. See Proposition 7. □

We shall conclude our study with the following, uneasy

Conjecture 26 *We have either* $\mathcal{K}(\mathcal{B}, R) = \mathrm{O}(4)$*, or* $\mathcal{K}(\mathcal{B}, R) = \mathrm{Sp}(1)$*, or* $\mathcal{K}(\mathcal{B}, R)$ *contains* $\mathrm{SO}(2) \times \mathrm{SO}(2)$ *and it is not a group, or* $\mathcal{K}(\mathcal{B}, R) = \mathcal{H}_5$*.*

Acknowledgements The authors thank to Professor K. Sekigawa for his inspiring personal communication. The first author was supported by the Institutional Support for the Development of the Research Organization, University of Hradec Králové. The second author was supported by the grant GAČR 201/11/0356.

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