Bilinear Strichartz estimates and applications to the cubic nonlinear Schrödinger equation in two space dimensions*

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(Received March 11, 2008; Revised July 14, 2008)

Abstract. The initial value problem for the defocusing cubic nonlinear Schrödinger equation on \( \mathbb{R}^2 \) is locally well-posed in \( H^s \) for \( s \geq 0 \). The \( L^2 \)-space norm is invariant under rescaling to the equation, then the critical regularity is \( s = 0 \). In this note, we prove the global well-posedness in \( H^s \) for all \( s > 1/2 \). The proof uses the almost conservation approach by adding additional (non-resonant) correction terms to the original almost conserved energy.

Key words: Strichartz estimate, nonlinear Schrödinger equation, global well-posedness.

1. Introduction

This note concerns with the initial value problem (IVP) for the cubic nonlinear Schrödinger equation in \( \mathbb{R}^{1+2} \)

\[
\begin{aligned}
  i\partial_t u(t, x) + \Delta u(t, x) &= |u(t, x)|^2 u(t, x), \quad (t, x) \in \mathbb{R}^{1+2} \\
  u(0, x) &= u_0(x) \in H^s(\mathbb{R}^2),
\end{aligned}
\]

(1.1)

where \( H^s(\mathbb{R}^2) \) denotes the inhomogeneous Sobolev space. In general, the conservation laws of \( L^2 \)-mass and \( H^1 \)-energy can be used to obtain the global well-posedness results in \( L^2 \) and \( H^1 \) spaces. We will be interested in the global-in-time well-posedness of (1.1) for low-regularity \( s \) below the energy regularity.

The equation (1.1) has the \( L^2 \)-mass conservation law

\[
\int_{\mathbb{R}^2} |u(t, x)|^2 dx = \int_{\mathbb{R}^2} |u_0(x)|^2 dx,
\]

(1.2)

2000 Mathematics Subject Classification : 35Q55.

*This is joint work with James Colliander, Markus Keel, Gigliola Staffilani and Terence Tao [8]. This note summarizes the result the author presented at the Nonlinear Wave Equations at Hokkaido University.
and the (total) energy conservation law
\[ E[u(t)] := \int_{\mathbb{R}^2} \frac{1}{2} |\nabla u(t, x)|^2 + \frac{1}{4} |u(t, x)|^4 dx = E[u_0]. \quad (1.3) \]
It is known that the IVP (1.1) is locally well-posed when \( s \geq 0 \), and the time-interval of existence of solution can be obtained in the term of \( H^s \) norm of the data when \( s > 0 \) (cf. [4, 14])\(^1\). Moreover, the solution-map \( u_0 \mapsto u(t) \) is continuous\(^2\) for \( s \geq 0 \), and not uniformly continuous for \( s < 0 \).

The \( L^2 \)-space is the critical space for (1.1) with respect to the scale invariant space under the scaling symmetry
\[ u(t, x) \mapsto \frac{1}{\lambda} u \left( \frac{t}{\lambda^2}, \frac{x}{\lambda} \right), \quad \lambda > 0 \quad (1.4) \]
or the Galilean invariant space under the Galilean symmetry
\[ u(t, x) \mapsto e^{ix \cdot v + it(|v|^2/2)} u(t, x - vt), \quad v \in \mathbb{R}^2. \]

From the conservation laws (1.2)–(1.3) (and Sobolev inequality), we obtain the time-global a priori estimate for solutions in \( H^s \) for \( s = 0 \) or \( s = 1 \) of the form
\[ \sup_{|t| \leq T} \| u(t) \|_{H^s} \leq C(s, \| u_0 \|_{H^s}, T) \quad (1.5) \]
for all \( T > 0 \). This form of the a priori bound in conjunction with the proof of the local existence theory (in particular the time interval to guarantee the existence of solution depends on the \( H^s \)-norm of data) can be used to prove the global well-posedness in \( H^s \) for \( s \geq 1 \). The a priori estimate (1.5) holds for \( s = 0 \), but the lack of the \( L^2 \)-local well-posedness theorem cannot immediately prove the global well-posedness result in \( L^2 \) (If data are small, the global well-posedness holds in \( L^2 \) including scattering result).

The first breakthrough to establish the \( H^s \)-global well-posedness for fractional exponent \( s \) of \( H^s \) has been developed by the Fourier truncation method of J. Bourgain \[2\] who obtained for \( s > 3/5 \). This result was improved by the Almost conserved quantities \[7\], which obtained the estimate (1.5) for \( s > 4/7 \). These two methods use a low-frequency/high-frequency

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\(^1\)When \( s = 0 \), the time-interval of existence of solution depends upon the profile of data.

\(^2\)When \( s > 0 \), the \( H^s \) well-posedness is shown by the contraction mapping theorem. Then the solution-map is analytic.
decomposition approach, but estimate the nonlinear interactions (low-high energy cascade) in different ways.

**Fourier truncation method.** With a cut-off frequency $|\xi| = N$, we assess the low-frequency component in $|\xi| \leq N$ and the high-frequency component in $|\xi| > N$, respectively. Roughly, if the solution is decomposed into three components: low frequencies nonlinearly (via original equation (1.1)), high frequencies nonlinearly (via original equation (1.1)) and the low-high frequencies interaction nonlinearly (coupled equation of low-high frequencies). The low frequencies solution conserves the $H^1$-energy (1.3), but this is large. One observes that the high frequencies solution can be approximated to evolve linearly. One proves that the low-high (high-high also) frequency interactions are small error in $H^1$, under certain smoothing property, compared to the low-low frequency interactions, provided choosing $N$. On the other hand, the almost conserved quantities proceed slightly different with the method.

**Almost conserved quantities.** This method uses the modified energy $E_N[u(t)] = E[I_N u(t)]$, where $I = I_N$ is a Fourier multiplier operator mapping from $H^s$ to $H^1$ (defined in Section 2.1). More precisely, $I = \text{identy}$ for low frequency, while $I = N^{1-s} \nabla^{s-1}$ for high frequency. If $N$ large, the modified energy $E_N[(u(t))]$ is qualifiedly equal to the original energy $E[u(t)]$. The low frequency interaction is estimated in $H^1$ energy space. The low-high (high-high also) frequency interaction is estimated with approximately conserved energy $E_N$ in $I_N H^s$. Thus the multiplier operator $I_N$ has an advantage of improving the estimate developed in [7] for the low-high frequency nonlinear interactions.

In this note we improve the result of [7] and present the following theorem.

**Theorem 1** Let $s > 1/2$. Then the initial value problem (1.1) is globally well-posed in $H^s$. More specifically, for any $u_0 \in H^s$, there exists a unique global solution $u(t)$ to (1.1) in $C_t(\mathbb{R}; H^s_x)$. Furthermore, the a priori bound (1.5) holds.

The proof relies on the modification of the almost conserved quantities used in [7] by adding resonant correction terms.
Remark 1  Theorem 1 holds for the focusing cubic nonlinear Schrödinger equation (replacing the sign of nonlinearity), assuming the smallness of the \(L^2\)-norm of the initial data \(\|u_0\|_{L^2} < \|Q\|_{L^2}\), where \(Q\) is the positive solution of \(\Delta Q - Q = -Q^3\) (grand state solution for the focusing nonlinear Schrödinger equation).

Remark 2  Fang and Grillakis obtained the global well-posedness at \(s = 1/2\) by using the interaction Morawetz estimate [11], and Colliander, Grillakis and Tzirakis [5] improved this for \(s > 2/5\) by combining the Morawetz estimate with the almost conserved quantities. In more recently, Klip, Tao and Visan [12] obtained global well-posedness and scattering for all \(s \geq 0\), though radial data. But Theorem 1 (in particular Theorem 2) seems interesting, because the angularly constrained Strichartz estimate (Corollary 1) in conjunction with [5, 11] may improve the global well-posedness for \(s > 4/13\) without radial condition on the initial data.

Open problem  It is conjectured that (1.1) is globally well-posed and scatters to free solution for all data in \(L^2\). This conjecture still remains open.

2. Sketch of the proof of Theorem 1

The strategy of the almost conserved quantities is as follows: First fix an arbitrary time interval \([0, T]\). Let \(E_N[u(t)]\) be a new energy for solutions in \(H^s\) depending on a parameter \(N \gg 1\) and take the rescaling. We prove again the local well-posedness result in the space associated to \(E_N[(u(t)]\) on time intervals of length \(\delta \approx 1\). Finally, we perform the iteration on the time interval \([0, T]\) to derive the a priori estimate of solutions with rescaling. How is it that our argument is successful? The variant of the energy \(E_N\) is very slowly in \(t\). In particular, the energy \(E_N\) is almost conserved to evolve of (1.1). For Theorem 1, we use a slight variant \(\hat{E}[u(t)] = \hat{E}_N[u(t)]\) of \(E_N[u(t)]\).

2.1. Almost conserved quantity

An almost conserved quantity is defined as follows: Let \(N \gg 1\) and

\[
\hat{I} u(\xi) = \hat{I}_N u(\xi) = m(\xi) \hat{u}(\xi),
\]

where \(m(\xi)\) is an even \(C^\infty\)-monotone function which equals to 1 for \(|\xi| < N\) and equals to \((|\xi|/N)^{s-1}\) for \(|\xi| > 2N\). We define
\[ E_N[u(t)] = E[Iu(t)]. \] (2.1)

This quantity is almost conserved to evolve the solution of (1.1) in the following sense:

\[
\frac{d}{dt} E_N[u(t)] = -2 \Re \int_{\mathbb{R}^2} I\partial_t u(I|u|^2 u - |Iu|^2 Iu) dx = O(N^{-\alpha}),
\] (2.2)

for some \( \alpha > 0 \) (\( \alpha = 3/2 - \varepsilon \) is obtained in [7]). With \( E_N \), we obtain the a priori estimate (1.5) for \( s > 4/7 \).

### 2.2. Resonant correction terms

Improving the error term \( N^{-\alpha} \) in (2.2), we try to remove the biquadratic term in (2.2). At the present, we use the following modified energy functional \( \tilde{E}[u(t)] \):

\[
\tilde{E}[u(t)] = \Lambda_2(\sigma_2; u) + \Lambda_4(\tilde{\sigma}_4; u)
\]

where

\[
\Lambda_k(\sigma; u) = \int_{\xi_1 + \ldots + \xi_k = 0} \sigma(\xi_1, \ldots, \xi_k) \hat{u}(\xi_1) \cdots \hat{u}(\xi_k),
\]

\[
\sigma_2 = -\frac{1}{2} \xi_1 m_1 \xi_2 m_2,
\]

\[
\tilde{\sigma}_4 = \frac{|\xi_1|^2 m_1^2 - |\xi_2|^2 m_2^2 + |\xi_3|^2 m_3^2 - |\xi_4|^2 m_4^2}{4(|\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2)} 1_{\Omega_{nr}},
\]

\( \Omega_{nr} = \{(\xi_1, \ldots, \xi_4): \max |\xi_k| \leq N \text{ or } |\cos \angle(\xi_{12}, \xi_{14})| \geq \theta > 0\} \),

\( (\xi_{ij} = \xi_i + \xi_j \text{ etc} ) \) \( 1_{\Omega_{nr}} \) is the characteristic function on \( \Omega_{nr} \), and \( m_k = m(\xi_k). \) \( \theta = \theta(N) > 0 \) is defined later depending on \( N \) (Section 2.4).

**Remark 3** With the above functions, we can write the first generation of the modified energy \( E_N[u(t)] \) as follows:

\[
E_N[u(t)] = \Lambda_2(\sigma_2; u) + \Lambda_4(\sigma_4; u),
\]

where

\[
\sigma_4 = \frac{1}{4} m_1 m_2 m_3 m_4.
\]

**Remark 4** The resonant sets

\[
\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0,
\]
$0 = |\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2 = 2 \xi_{12} \cdot \xi_{14}$.

($\xi_{12}$ and $\xi_{14}$ are almost orthogonal) are almost canceled from the biquadratic form of $\Lambda_4(\sigma_4; u)$ in the following sense

$$\frac{d}{dt} \tilde{E}[u(t)] = \Lambda_4(\tilde{\sigma}_4; u) + \Lambda_6(\tilde{\sigma}_6; u) \tag{2.3}$$

where

$$\tilde{\sigma}_4 = (|\xi_1|^2 m_1^2 - |\xi_2|^2 m_2^2 + |\xi_3|^2 m_3^2 - |\xi_4|^2 m_4^2)1_{\Omega_r},$$

$$\Omega_r = \{(\xi_1, \ldots, \xi_4) : \max|\xi_k| > N \text{ and } \cos \angle(\xi_{12}, \xi_{14}) < \theta\}$$

(we skip the details for the symbol $\tilde{\sigma}_6$ in this note, because the biquadratic term is leading). Exploiting the presence of resonance condition $\Omega_r$ is our improvement of the previous work in [7].

The interest of the resonance condition lies in the following estimate.

**Theorem 2** Angularly refined bilinear Strichartz estimate \textit{Let $0 < N_1 < N_2$ and $\theta \in (0, 1/100)$. For $\phi_1, \phi_2 \in L^2$ with Fourier frequencies $N_1, N_2$, respectively, we have}

$$\|F\|_{L^2_{t,x}} \lesssim \min\left\{\theta, \frac{N_1}{N_2}\right\}^{1/2} \|\phi_1\|_{L^2} \|\phi_2\|_{L^2},$$

where

$$F(t, x) = \int_{\xi_1 + \xi_2 = 0} e^{ix(\xi_1 + \xi_2) t} 1_{|\cos \angle(\xi_1, \xi_2)| \leq \theta e^{it\Delta} \phi_1(\xi_1) e^{it\Delta} \phi_2(\xi_2)}.$$

The above estimate without angularly constrained was already obtained in [2]. The proof of Theorem 2 follows an argument in [2] under the additional restriction on the angle between interacting frequencies.

We recall the Fourier restriction norm space $X_{s,b}[0, T]$ with the following norm [1]:

$$\|f\|_{X_{s,b}[0, T]} = \inf \{\|g\|_{X_{s,b}} \mid f = g \text{ on } (t, x) \in [0, T] \times \mathbb{R}^2\}$$

where

$$\|g\|_{X_{s,b}}^2 = \int_{\mathbb{R}^3} (1 + |\tau + |\xi|^2|)^{2b}(1 + |\xi|)^{2s} |\tilde{g}(\tau, \xi)|^2 d\xi d\tau.$$

The following corollary immediately follows from Theorem 2.
Corollary 1 Let $0 < N_1 < N_2$ and $\theta \in (0, 1/100)$. For $u_1, u_2 \in X_{0,1/2+\epsilon}$ such that
\[
\text{supp } \widehat{u_1}(t, \xi) = \{ \xi | \sim N_1 \}, \text{ supp } \widehat{u_2}(t, \xi) = \{ \xi | \sim N_2 \},
\]
and $|\cos \angle (\xi_1, \xi_2)| \leq \theta$ for $\xi_1 \in \text{supp } \widehat{u_1}(t, \xi)$ and $\xi_2 \in \text{supp } \widehat{u_2}(t, \xi)$, we have
\[
\|u_1 u_2\|_{L^2(\mathbb{R}^{1+2})} \lesssim \min \left\{ \theta, \frac{N_1}{N_2} \right\}^{1/2} \|u_1\|_{X_{0,1/2+\epsilon}} \|u_2\|_{X_{0,1/2+\epsilon}}. \tag{2.4}
\]

2.3. Local well-posedness in $IH^s$-space

In this section we prove the local well-posedness of the initial value problem obtained by acting on (1.1) with the operator $I$
\[
\begin{cases}
iIu_t(t, x) + \Delta Iu(t, x) = I(|u(t, x)|^2 u(t, x)), \\
Iu(0, x) = Iu_0(x).
\end{cases}
\tag{2.5}
\]

We still have the following local well-posedness theorem (cf. [4, 14, 7]).

Lemma 1 (Modified Local well-posedness) Let $s > 0$. The Cauchy problem (2.5) is locally well-posed on $[0, T_0]$, $T_0 = T_0(\|Iu_0\|_{H^1})$ with solution $u(t)$ such that
\[
Iu \in C([0, T]; H^1), \quad \|Iu\|_{X_{1,1/2+\epsilon}[0,T_0]} \lesssim (\|Iu_0\|_{H^1}).
\]

Next we give that $\tilde{E}[u(t)]$ controls data size as follows:

Lemma 2 Let $u(t) \in H^s (s > 1/2)$ be a solution to (1.1). Then
\[
\|Iu(t)\|^2_{H^1} \leq \tilde{E}[u(t)] + \frac{c}{\theta N^2} \|Iu(t)\|^2_{H^1} \|Iu(t)\|^2_{H^1},
\]
where $\theta$ is given by (2.3).

The proof of Lemma 2 is essentially similar to [7]. Therefore we omit details.

2.4. $\tilde{E}[u(t)]$ obeys the almost conservation law

Lemma 3 (Almost conservation) Let $u(t) \in H^s (s > 1/2)$ be a solution to (1.1). For $t \geq 0$, we have
\[
\tilde{E}[u(t)] \leq \tilde{E}[u(0)] + \left( \frac{1}{N^{2-\epsilon}} + \frac{\theta^{1/2}}{N^{3/2-\epsilon}} + \frac{1}{\theta N^{3-\epsilon}} \right) C(\|Iu\|_{X_{1,1/2+\epsilon}[0,t]}).
\]

Note that the choice $\theta = 1/N$ produces the pre-factor $cN^{-2+\epsilon}$. 
A brief outline of the proof of Lemma 3. By (2.3), we have
\[
\bar{E}[u(t)] - \bar{E}[u(0)] = \int_0^t \Lambda_4(\bar{\sigma}_4; u) + \Lambda_6(\bar{\sigma}_6; u) \, ds = \mathcal{I}_1 + \mathcal{I}_2.
\]

We aim to show
\[
\mathcal{I}_1 + \mathcal{I}_2 \leq \left( \frac{1}{N^{2-\varepsilon}} + \frac{\theta^{1/2}}{N^{3/2-\varepsilon}} + \frac{1}{\theta N^{3-\varepsilon}} \right) C(\| I u \|_{X_{1,1/2+\varepsilon}[0,t]}).
\]  
(2.6)

We use the Littlewood-Paley decomposition and break \( u \) into \( u = \sum_N u_N \) where \( \text{supp} \, \hat{u}(\xi) = \{ |\xi| \sim N \} \).

We consider (2.6) for the term \( \mathcal{I}_1 \) and provide proofs for some special cases: \( N_1 \sim N_2 \geq N, \ N_3 \gg N_4 \) under the resonant condition \( |\cos \angle(\xi_{12}, \xi_{14})| \leq \theta \). We need the following calculations
\[
|\bar{\sigma}_4| \leq c(m(N_1)^2 N_1 N_3 \theta + m(N_3)^2 N_3^2),
\]
\[
|\cos \angle(\xi_1, \xi_3)| = |\cos \angle(\xi_{14}, \xi_{34})| + O\left( \frac{N_4}{N_3} \right) \leq \theta + O\left( \frac{N_4}{N_3} \right).
\]

Then taking \( u_{N_4} u_{N_4} \) in \( L^2 \), and \( u_{N_1} u_{N_3} \) in \( L^2 \), respectively, and using Corollary 1, we can estimate
\[
\leq c(m(N_1)^2 N_1 N_3 \theta + m(N_3)^2 N_3^2) \left( \frac{N_4}{N_2} \right)^{1/2} \left( \theta + \frac{N_4}{N_3} \right)^{1/2} \times \prod_{j=1}^4 \| u_{N_j} \|_{X_{1,1/2+\varepsilon}},
\]
this is bounded by
\[
\leq c \frac{m(N_1)^2 N_1 N_3 \theta + m(N_3)^2 N_3^2}{m(N_1)^2 N_1^2 N_3} \left( \frac{N_4}{N_2} \right)^{1/2} \left( \theta + \frac{N_4}{N_3} \right)^{1/2} \times \prod_{j=1}^4 \| I u_{N_j} \|_{X_{1,1/2+\varepsilon}}.
\]

Splitting into two cases: \( N_3 \geq \frac{N_4}{\theta} \) and \( N_3 < \frac{N_4}{\theta} \) and summing over \( N_1 \sim N_2 \geq N, \ N_3 \gg N_4 \), we have the bound (2.6). \( \blacksquare \)
2.5. Induction energy implies the a priori estimate (1.5)

Finally, we give a sketch of the induction argument that Lemmas 1, 2 and 3 imply Theorem 1.

Let \( u(t) \) be a smooth solution of (1.1). By (1.4), consider the rescaled solution

\[
u_\lambda(t, x) = \frac{1}{\lambda} u\left( \frac{t}{\lambda^2}, \frac{x}{\lambda} \right), \quad \lambda > 0.
\]

Fix an arbitrary time interval \([0, T]\). We prove Theorem 1, if we construct a rescaled solution \( u_\lambda \) on \([0, \lambda^2 T]\). As in [6, 7, 8] the idea is to reach time \( t = \lambda^2 T \) inductively using the energy estimates in Lemmas 2 and 3 and the local theory in Lemma 1. An easy computation shows that

\[
\|Iu_0\|_{H^1} \leq c\lambda^{-s} N^{1-s}\|u_0\|_{H^s} \ll 1
\]

provided we choose \( \lambda \gg \|u_0\|_{H^s}^{1/s} N^{(1-s)/s} \). From Lemmas 2 and 3, \( u(t) \) has the a priori estimate

\[
\|Iu(t)\|_{H^1} \leq \tilde{E}[u(t)] \leq \tilde{E}[u(0)] + cN^{-2+\varepsilon}C(\|Iu\|_{X_{1,1/2+\varepsilon}}[0,T]),
\]

(2.7)

We will show that by Lemma 1, \( \|Iu\|_{X_{1,1/2+\varepsilon}}[0,T_0] \leq C \) and \( T_0 = 1 \) whenever \( \|Iu_0\|_{H^1} \ll 1 \). Thus there is an increment in the energy of size at most \( N^{-2+\varepsilon} \), if \( \|Iu\|_{X_{1,1/2+\varepsilon}} \leq C \) in (2.7). Hence we want to ensure

\[
N^{2-\varepsilon} \geq c\lambda^2 T = N^{2(1-s)/s}TC(\|u_0\|_{H^s}),
\]

which is achieved for \( s > 1/2 \) by letting \( N = N(T) \) sufficiently large. Notice that

\[
\frac{1}{2}\|u(t)\|_{H^s}^2 \leq \tilde{E}[u(t)] + c\|u(t)\|_{L^2}^2.
\]

This proves Theorem 1, using the \( L^2 \)-conservation law (1.2).

\[\square\]

References


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