Basis properties and complements of complex exponential systems

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(Received September 15, 2005; Revised December 12, 2005)

Abstract. In this note, we show that some families of complex exponentials are either Riesz sequences or not basic sequences in $L^2[-\pi, \pi]$. Besides, we show that every incomplete complex exponential system satisfying some condition can be complemented up to a complete and minimal system of complex exponentials in $L^2[-\pi, \pi]$.

Key words: basis, Riesz basis, Riesz sequence, complete and minimal sequence.

1. Introduction

Let $\lambda = \{\lambda_n\}, -\infty < n < \infty$, be a sequence of distinct complex numbers, then a system $e(\lambda) \equiv \{e^{i\lambda_n t}\}$ is said to be a basis for $L^2[-\pi, \pi]$ if any function $f(t)$ in $L^2[-\pi, \pi]$ has a unique expansion $f(t) = \sum_n c_n e^{i\lambda_n t}$ (in the mean) for some sequence $\{c_n\}$. Also, $e(\lambda)$ is said to be a basic sequence if it is a basis of the closure of the space spanned by the distinct elements $e^{i\lambda_n t}$. Next $e(\lambda)$ is said to be a Riesz basis if there exists an isomorphism $T: L^2[-\pi, \pi] \to L^2[-\pi, \pi]$ and $T(e^{int}) = e^{i\lambda_n t}$ for any $n$. Moreover, $e(\lambda)$ is said to be a Riesz sequence if it is a Riesz basis of the closure of the space spanned by the distinct elements $e^{i\lambda_n t}$. $e(\lambda)$ is said to be complete in $L^2[-\pi, \pi]$ if the linear subspace spanned by the distinct elements $e^{i\lambda_n t}$ is dense in $L^2[-\pi, \pi]$. And $e(\lambda)$ is said to be minimal in $L^2[-\pi, \pi]$ if each element of $e(\lambda)$ lies outside the closed linear span of the others. Obviously, we see that if $e(\lambda)$ is a Riesz basis, then it is a basis and if it is a basis, then it is complete and minimal. We say the system $\{e^{i\lambda_n t}\}$
has *excess* \( N \) if it remains complete and becomes minimal when \( N \) terms \( e^{i\lambda_n t} \) are removed and we define

\[
E(\lambda) = N.
\]

Conversely we define the excess

\[
E(\lambda) = -N
\]

if it becomes complete and minimal when \( N \) terms \( e^{i\mu_1 t}, \ldots, e^{i\mu_N t} \) are adjoined. By convention we define \( E(\lambda) = \infty \) if arbitrarily many terms can be removed without losing completeness and \( E(\lambda) = -\infty \) if arbitrarily many terms can be adjoined without getting completeness. It is obvious that \( \{e^{i\lambda_n t}\} \) is to be complete and minimal if and only if \( E(\lambda) = 0 \).

We refer to N. Levinson [L], R.M. Young [Y4] and R.M. Redheffer [R] on the theory of nonharmonic Fourier series which we take up in this note.

R.M. Young showed in the proof of [Y2, Theorem 2] that if

\[
\lambda_n = \begin{cases} 
 n - \frac{1}{4}, & n > 0, \\
 n + \frac{1}{4}, & n < 0,
\end{cases}
\tag{1.1}
\]

then \( e(\lambda) \) was not a basis. Besides he showed in [Y3, Theorem 2] that if

\[
\mu_n = \begin{cases} 
 n + \frac{1}{4}, & n > 0, \\
 0, & n = 0, \\
n - \frac{1}{4}, & n < 0,
\end{cases}
\tag{1.2}
\]

then \( e(\mu) \) was not also a basis. In this note, we first show that if

\[
\lambda_n = \begin{cases} 
 n - \alpha, & n > 0, \\
 n + \alpha, & n < 0,
\end{cases}
\]

and

\[
\mu_n = \begin{cases} 
 n + \alpha, & n > 0, \\
 0, & n = 0, \\
n - \alpha, & n < 0,
\end{cases}
\]
then \(e(\lambda)\) and \(e(\mu)\) are either Riesz sequences or not basic sequences in \(L^2[-\pi, \pi]\) for \(0 < \alpha < 1\).

Next let
\[
\lambda_n = \begin{cases} 
    n - \alpha_n, & n > 0, \\
    n + \alpha_n, & n < 0 
\end{cases}
\]
for \(0 < \alpha_n < 1\), then we consider whether \(e(\lambda)\) is a Riesz basis or not, and moreover it is a basis or not. One of the problems is whether \(e(\lambda)\) is a basis or not in the case of which
\[
\varepsilon_n \to 0 \quad \text{as} \quad n \to \pm \infty \quad \text{and} \quad \sum_n |\varepsilon_n| = \infty
\]
for \(\alpha_n = 3/4 + \varepsilon_n\).

In this note, we need the following “stability results”.

**Theorem A** (see [Y4, p. 161, Corollary]) If \(e(\lambda)\) is a Riesz basis for \(L^2[-\pi, \pi]\), then there is a positive constant \(L\) with the property that \(e(\mu)\) is also a Riesz basis for \(L^2[-\pi, \pi]\) whenever \(|\lambda_n - \mu_n| \leq L\) for every \(n\).

**Theorem B** (see [Y4, p. 165, Prob. 2]) Let \(e(\lambda)\) be a basis for \(L^2[-\pi, \pi]\) and suppose that \(\sup_n |\Im \lambda_n| < \infty\). If \(\mu = \{\mu_n\}\) satisfies
\[
\sum_n |\lambda_n - \mu_n| < \infty,
\]
then, \(e(\mu)\) is also a basis for \(L^2[-\pi, \pi]\).

The following result follows from Theorem B immediately. We see also Lemma II.4.11 of S.A. Avdonin and S.A. Ivanov [AI] about the same result.

**Corollary 1.1** We suppose that \(\sup_n |\Im \lambda_n| < \infty\) and \(e(\lambda)\) is a basis. If we replace finitely many points \(\lambda_n\) by the same number of points \(\mu_n \notin \{\lambda_n\}\), \(\mu_n \neq \mu_m, n \neq m\), then the basis property of \(e(\lambda)\) is not violated. Consequently the same applies to any Riesz basis.

**Remark 1.1** Theorem A holds even if “Riesz sequence” that excess is finite is taken. So far as we know, it is unknown whether Theorem A holds or not if a Riesz basis is replaced with a basis. However, it is also unknown whether such a basis which is conditional exists or not.

In §4, we show that every incomplete complex exponential system sat-
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satisfying some condition can be complemented up to a complete and minimal system of complex exponentials. It is unknown, so far as we know, whether every incomplete complex exponential system can be complemented up to a complete and minimal system of complex exponentials in \(L^2[-\pi, \pi]\) or not. This problem is originated in [Y1, Remark]. On the other hand, K. Seip has shown in [S, Theorem 2.8] that there exists a Riesz sequence of complex exponentials which cannot be complemented up to a Riesz basis. He has given a sequence

\[ e(\lambda) = \{e^{\pm i(n+\sqrt{n})t}\}_{n>1} \]

as an example of such a Riesz sequence.

And he raised the next question personally:

**Question** Can every Riesz sequence of complex exponentials be complemented up to a complete and minimal system of complex exponentials?

In this section, we show that it is possible for some families of complex exponential systems which include many Riesz sequences of \(E(\lambda) = -\infty\). Let \(e(\lambda)\) be a complex exponential system which has the excess \(E(\lambda) = -\infty\). Our method is to construct a sequence \(\mu = \{\mu_n\}\) such that \(\lambda \subset \mu\) and the system \(e(\mu)\) has a finite excess. If we can construct such a sequence \(\mu\), then we see that the system \(e(\lambda)\) can be complemented up to a complete and minimal system of complex exponentials in \(L^2[-\pi, \pi]\). For this purpose, we use the next theorem:

**Theorem C** ([R, Theorem 47]) For \(-\infty < n < \infty\), let \(\lambda \equiv \{\lambda_n\}\) be a sequence of complex numbers satisfying \(|\lambda_n - n| \leq h\) where \(h\) is a positive constant. Then \(E(\lambda)\) satisfies

\[-\left(4h + \frac{1}{2}\right) < E(\lambda) \leq 4h + \frac{1}{2}\]

**2. Basis properties of complex exponential systems**

We first consider the system \(e(\lambda)\),

\[ \lambda_n = \begin{cases} n - \alpha, & n > 0, \\ n + \alpha, & n < 0, \end{cases} \quad (2.1) \]

for \(0 < \alpha < 1\). We see from Kadec’s 1/4-theorem (M.I. Kadec, 1964; see [Y4, p. 36]) that \(e(\lambda)\) is a Riesz sequence for \(0 < \alpha < 1/4\). It has been shown in
[Y2, Theorem 2] that $e(\lambda)$ is not a basis for $L^2[-\pi, \pi]$ for $\alpha = 1/4$. Besides, it has been also known that $e(\lambda)$ is a Riesz basis for $L^2[-\pi, \pi]$ for $1/4 < \alpha < 3/4$ by using the isometric isomorphism
\[
\phi(t) \mapsto e^{it/2}\phi(t)
\]
on $L^2[-\pi, \pi]$ and Kadec’s 1/4-theorem.

**Proposition 2.1**  Let $\lambda = \{\lambda_n\}$ be a sequence given by (2.1).
(i) Let $\alpha = 3/4$. If we remove any element $\lambda_{n_0}$ in $\lambda$, then $e(\lambda')$ for $\lambda' = \lambda - \{\lambda_{n_0}\}$ is complete and minimal, but it is not a basis for $L^2[-\pi, \pi]$;
(ii) $e(\lambda')$ of (i) is a Riesz basis for $L^2[-\pi, \pi]$ for $3/4 < \alpha < 1$.

**Proof.** We see that $e(\lambda')$ is complete and minimal by [N, Theorem 1.1]. But if we write
\[
n - \frac{3}{4} = (n - 1) + \frac{1}{4}, \quad -n + \frac{3}{4} = -(n - 1) - \frac{1}{4}
\]
for $n \geq 1$, then we see that $e(\lambda')$ is not a basis for $L^2[-\pi, \pi]$ because $e(\mu)$, where the $\mu_n$ are given by (1.2), is not a basis. This prove (i).

Now if we write
\[
n - \alpha = (n - 1) + (1 - \alpha), \quad -n + \alpha = -(n - 1) - (1 - \alpha)
\]
for $n \geq 1$, then (ii) is trivial by Kadec’s 1/4-theorem. $\square$

Next we consider the system $e(\mu)$,
\[
\mu_n = \begin{cases} 
  n + \alpha, & n > 0, \\
  0, & n = 0, \\
  n - \alpha, & n < 0,
\end{cases} \tag{2.2}
\]
for $0 < \alpha < 1$.

It has already been known by Kadec’s 1/4-theorem that $e(\mu)$ is a Riesz basis for $0 < \alpha < 1/4$. It has also been shown in [Y3, Theorem 2] that $e(\mu)$ is not a basis for $L^2[-\pi, \pi]$ for $\alpha = 1/4$.

**Proposition 2.2**  Let $\mu = \{\mu_n\}$ be a sequence given by (2.2).
(i) $e(\mu)$ is not a basic sequence for $\alpha = 3/4$;
(ii) $e(\mu)$ is a Riesz sequence for $1/4 < \alpha < 3/4$ or $3/4 < \alpha < 1$. 

Proof. If we write
\[ n + \frac{3}{4} = (n + 1) - \frac{1}{4}, \quad -n - \frac{3}{4} = -(n + 1) + \frac{1}{4} \]
for \( n \geq 1 \), then (i) is an immediate consequence from the fact that \( e(\lambda) \), where the \( \lambda_n \) are given by (1.1), is not a basis for \( L^2[-\pi, \pi] \). Next we write
\[ n + \alpha = (n + 1) - (1 - \alpha), \quad -n - \alpha = -(n + 1) + (1 - \alpha) \]  
for \( n \geq 1 \). We see that \( e(\mu) \) is a Riesz sequence from (2.3) and the known result which \( e(\lambda) \) given by (2.1) is a Riesz basis for \( 1/4 < \alpha < 3/4 \). Moreover, it is a Riesz sequence by (2.3) and Kadec’s 1/4-theorem for \( 3/4 < \alpha < 1 \). □

From the above results, we have obtained the following results:

**Corollary 2.1** Let \( e(\gamma) \) be a system given by (2.1) or (2.2), then \( e(\gamma) \) is either a Riesz sequence or not a basic sequence in \( L^2[-\pi, \pi] \).

Now we next consider the system \( e(\lambda) \),
\[ \lambda_n = \begin{cases} n - \alpha_n, & n > 0, \\ n + \alpha_n, & n < 0, \end{cases} \quad (2.4) \]
for \( 0 < \alpha_n < 1 \). The cases of \( \sup_n \alpha_n < 1/4 \) and \( 1/4 < \inf_n \alpha_n < 3/4 \), \( 3/4 < \inf_n \alpha_n \) are trivial by Kadec’s 1/4-theorem, and so we deal with the case which the numbers \( \alpha_n \) behave the neighborhood of \( 1/4 \) or \( 3/4 \).

**Theorem 2.1** Let \( \alpha_n = 3/4 + \varepsilon_n \) or \( \alpha_n = 1/4 + \varepsilon_n \). Then we obtain the following results for \( e(\lambda) \) given by (2.4):
(1) If \( \varepsilon_n \to 0 \) as \( n \to \pm\infty \), then \( e(\lambda) \) is not a Riesz basis for \( L^2[-\pi, \pi] \).
(2) Furthermore, if \( \sum_n |\varepsilon_n| < \infty \), then \( e(\lambda) \) is not a basis for \( L^2[-\pi, \pi] \).

Proof. First we consider the case of \( \alpha_n = 3/4 + \varepsilon_n \) in (2.4). Then
\[ \lambda_n = \begin{cases} n - \frac{3}{4} - \varepsilon_n, & n > 0, \\ n + \frac{3}{4} + \varepsilon_n, & n < 0. \end{cases} \]
Now, if we take
\[ \gamma_n = \begin{cases} n - \frac{3}{4}, & n > 0, \\ n + \frac{3}{4}, & n < 0, \end{cases} \]

\( e(\gamma) \) is not a basis for \( L^2[-\pi, \pi] \) by (i) of Proposition 2.1.

We suppose \( \varepsilon_n \to 0 \) as \( n \to \pm\infty \). We refer to [RY, p. 107, Corollary] about the next argument. If \( e(\lambda) \) is a Riesz basis for \( L^2[-\pi, \pi] \), then there exists a positive constant \( L \) by Theorem A such that if

\[ |\lambda_n - \delta_n| \leq L \text{ for } \forall n, \]

\( e(\delta) \) is also a Riesz basis for \( L^2[-\pi, \pi] \). By the hypothesis, we can choose a positive integer \( n_0 \) such that

\[ |\lambda_n - \gamma_n| = |\varepsilon_n| \leq L \text{ for } \forall |n| \geq n_0. \]

Hence,

\[ \{ e^{i\lambda_n t} \}_{|n| < n_0} \cup \{ e^{i\gamma_n t} \}_{|n| \geq n_0} \]

is a Riesz basis for \( L^2[-\pi, \pi] \). Consequently, by Corollary 1.1, \( e(\gamma) \) is also a Riesz basis for \( L^2[-\pi, \pi] \). This contradicts, hence \( e(\lambda) \) is not a Riesz basis for \( L^2[-\pi, \pi] \).

Next we suppose \( \sum_n |\varepsilon_n| < \infty \). If \( e(\lambda) \) is a basis for \( L^2[-\pi, \pi] \), then \( e(\gamma) \) is also a basis for \( L^2[-\pi, \pi] \) by Theorem B. This contradicts, hence \( e(\lambda) \) is not a basis.

Second we consider the case of \( \alpha_n = 1/4 + \varepsilon_n \) in (2.4). Then

\[ \lambda_n = \begin{cases} n - \frac{1}{4} - \varepsilon_n, & n > 0, \\ n + \frac{1}{4} + \varepsilon_n, & n < 0. \end{cases} \]

We suppose that \( e(\lambda) \) is a Riesz basis for \( L^2[-\pi, \pi] \). Considering the isometric isomorphism

\[ \phi(t) \mapsto e^{it/2}\phi(t), \]

it follows that \( e(\lambda^{(1)}) \) is also a Riesz basis for \( L^2[-\pi, \pi] \), where

\[ \lambda_n^{(1)} = \begin{cases} n + \frac{1}{4} - \varepsilon_n, & n > 0, \\ n + \frac{3}{4} + \varepsilon_n, & n < 0. \end{cases} \]
Moreover, we rewrite
\[
\lambda_n^{(1)} = (n + 1) - \frac{1}{4} + \varepsilon_n, \quad n < 0,
\]
and if we substitute 0 for \(\lambda_n^{(1)} - 1\), we see that \(e(\lambda_n^{(2)})\) is also a Riesz basis for \(L^2[-\pi, \pi]\), where
\[
\lambda_n^{(2)} = \begin{cases} 
  n + \frac{1}{4} - \varepsilon_n, & n > 0, \\
  0, & n = 0, \\
  n - \frac{1}{4} + \varepsilon_{n-1}, & n < 0,
\end{cases}
\]
by Corollary 1.1. By the way, we know that \(e(\mu)\) is not a basis for the sequence \(\{\mu_n\}\) given by (1.2). Following the argument of the proof of the case \(\alpha_n = 3/4 + \varepsilon_n\) if \(\varepsilon_n \rightarrow 0\) as \(n \rightarrow \pm\infty\), we see \(e(\mu)\) is a Riesz basis for \(L^2[-\pi, \pi]\). This contradicts, hence \(e(\lambda)\) is not a Riesz basis for \(L^2[-\pi, \pi]\).

Next we suppose \(\sum_n |\varepsilon_n| < \infty\). If \(e(\lambda)\) is a basis, then \(e(\lambda_n^{(2)})\) is also a basis. Now, by the same argument as the one used in the case \(\alpha_n = 3/4 + \varepsilon_n\), it follows that \(e(\mu)\) is also a basis. This contradicts too, hence \(e(\lambda)\) is not a basis.

3. Some problems

From the examination until now, we have some problems. We suppose that
\[
\varepsilon_n \rightarrow 0 \text{ as } n \rightarrow \pm\infty \quad \text{and} \quad \sum_n |\varepsilon_n| = \infty
\]
for \(\alpha_n = 3/4 + \varepsilon_n\) in (2.4). Then, we have some questions, i.e., does \(e(\lambda)\) become a basis for \(L^2[-\pi, \pi]\) or a basic sequence?

We write
\[
\lambda_n = \begin{cases} 
  n - \alpha_n = (n - 1) + \frac{1}{4} - \varepsilon_n, & n > 0, \\
  n + \alpha_n = (n + 1) - \frac{1}{4} + \varepsilon_n, & n < 0.
\end{cases}
\]
And let \(\lambda' = \{\lambda'_n\}\).
\[ \lambda'_n = \begin{cases} n + \frac{1}{4} - \varepsilon_n, & n > 0, \\ 0, & n = 0, \\ -\lambda'_{-n}, & n < 0, \end{cases} \tag{3.1} \]

then, we have \( E(\lambda - \{\lambda_1\}) = E(\lambda') \) and the basis property of \( e(\lambda - \{\lambda_1\}) \) is same as the one of \( e(\lambda') \) by Corollary 1.1.

1. The case that \( e(\lambda') \) is complete.

As

\[ E(\lambda - \{\lambda_1\}) = E(\lambda') \geq 0, \]

we have \( E(\lambda) \geq 1 \), hence \( e(\lambda) \) is not a basis for \( L^2[-\pi, \pi] \). This can happen if

\[ \sum_n \frac{|\varepsilon_n|}{|n| + 1} < \infty \tag{3.2} \]

by [R, p. 45].

2. The case that \( e(\lambda') \) is not complete.

Redheffer and Young have given the next example of the sequence \( \{\varepsilon_n\} \) which does not satisfy (3.2):

**Theorem D** (see [RY, Theorem 3]) Let

\[ \mu_n = \begin{cases} 0, & n = 0, \\ 1, & n = 1, \\ n + \frac{1}{4} + \frac{\beta}{\log n}, & n \geq 2, \\ -\mu_{-n}, & n < 0, \end{cases} \tag{3.3} \]

then \( e(\mu) \) is complete in \( L^2[-\pi, \pi] \) if \( 0 \leq \beta \leq 1/4 \) and not if \( \beta > 1/4 \).

More precisely, by [R, Theorem 47] and [FNR], we have \( E(\mu) = 0 \) for \( 0 \leq \beta \leq 1/4 \) and \( E(\mu) = -1 \) for \( \beta > 1/4 \).

**Problem 3.1** We raise the next problems:

(i) Is the system \( e(\mu) \) in Theorem D basis for \( 0 < \beta \leq 1/4 \)?

(ii) Is the system \( e(\mu) \) in Theorem D basic sequence for \( \beta > 1/4 \)?

Moreover we have the problem which is equivalent to the above problem (ii):
Let
\[ \gamma_1 = \frac{1}{4}, \quad \gamma_{-1} = -\gamma_1 \]
and
\[ \gamma_n = \begin{cases} 
  \frac{n - 3}{4} + \frac{\beta}{\log n}, & n \geq 2, \\
  \gamma_{-n}, & n \leq -2,
\end{cases} \]
then is the system \( e(\gamma) \) basis for \( L^2[-\pi, \pi] \) for \( \beta > 1/4 \)?

In (3.3), if we replace “\( n + 1/4 \)” with “\( n + 1/4 - \varepsilon \)”, where \( \varepsilon \) is any small positive number, then the above problems are trivial by Kadec’s 1/4-theorem.

4. Complements of complex exponential systems

In this section, we show that every incomplete complex exponential system satisfying some condition can be complemented up to a complete and minimal system of complex exponentials in \( L^2[-\pi, \pi] \). We have the following result.

**Theorem 4.1** Let \( \{\delta_n\} \) be a real sequence such that
\[ 1 \leq \delta_1, \quad \delta_n \leq \delta_{n+1} \]
and
\[ \lim_{n \to \infty} \delta_n = \infty. \]
If \( \lambda \equiv \{\lambda_n\} \) is a sequence where
\[ \lambda_0 = 0, \quad \lambda_n = n + \delta_n, \quad \lambda_{-n} = -\lambda_n \quad (n = 1, 2, \ldots), \]
then the system \( e(\lambda) \equiv \{e^{i\lambda_n t}\} \) has the excess \( E(\lambda) = -\infty \) in \( L^2[-\pi, \pi] \) and \( e(\lambda) \) can be complemented up to a complete and minimal system of complex exponentials in \( L^2[-\pi, \pi] \).

**Proof.** We may choose \( \mu = \{\mu_n\} \) such that \( \lambda \subset \mu \) and \( e(\mu) \equiv \{e^{i\mu_n t}\} \) is complete and it has a finite excess in \( L^2[-\pi, \pi] \).

Firstly, we choose a positive integer \( k_1 \) such that
\[ k_1 \leq \delta_1 < k_1 + 1. \]
Then we take
\[ \mu_0 = 0, \mu_1 = 1, \ldots, \mu_{k_1+1} = k_1 + 1. \]
Moreover, we define
\[ \mu_{k_1+2} = \begin{cases} k_1 + 2, & \delta_1 = k_1, \\ \lambda_1, & \delta_1 \neq k_1. \end{cases} \]
Generally we choose a positive integer \( k_j \) for \( j \geq 2 \) such that
\[ k_j \leq \delta_j < k_j + 1. \]
If \( k_j = k_{j-1} + \ell \) (\( 1 \leq \ell \leq k_j - k_{j-1} \)), we take
\[ \mu_{k_j+j} = k_j + j \]
\[ \mu_{k_j+(j-1)} = k_j + (j - 1) \]
\[ \vdots \]
\[ \mu_{k_j+(j+1-\ell)} = k_j + (j + 1 - \ell), \]
and
\[ \mu_{k_j+(j+1)} = \begin{cases} k_j + (j + 1), & \delta_j = k_j, \\ \lambda_j, & \delta_j \neq k_j. \end{cases} \]
Next if \( k_{j-1} = k_j \), we take
\[ \mu_{k_j+(j+1)} = \mu_{k_{j-1}+(j+1)} = \begin{cases} k_j + (j + 1), & \delta_j = k_j, \\ \lambda_j, & \delta_j \neq k_j. \end{cases} \]
Finally let \( \mu_{-n} = -\mu_n \). Thus we choose the sequence \( \mu = \{\mu_n\} \).

For \( t > 1 \), we denote by \( n(t) \) and \( n_1(t) \) the number of integers \( n \) inside the interval \( |x| \leq t \) and the number of points \( \mu_n \) inside the interval \( |x| \leq t \), respectively. From the definition of the sequence \( \{\mu_n\} \), we have
\[ n_1(t) \geq n(t), \]
and hence, we see by [Y4, pp. 99–100, Theorem 3, 4] that \( e(\mu) \) is complete in \( L^2[-\pi, \pi] \), i.e. \( E(\mu) \geq 0 \). Besides, since \( k_j \leq \delta_j < k_j + 1, \lambda_j = j + \delta_j \), we have
\[ k_j + j \leq \lambda_j < k_j + (j + 1). \]
Therefore we see that
\[ n - 1 \leq \mu_n \leq n \] for \( \forall n \geq 1 \)
hold. Since \( \mu_{-n} = -\mu_n \), the inequalities
\[ |\mu_n - n| \leq 1 \] for \( \forall n \)
hold. Applying Theorem C, we conclude that
\[ E(\mu) \leq 4, \]
consequently
\[ 0 \leq E(\mu) \leq 4. \]
Hence we can reduce \( e(\mu) \) to a complete and minimal system. Thus \( e(\lambda) \) has the excess \( E(\lambda) = -\infty \) in \( L^2[-\pi, \pi] \) and it can be complemented up to a complete and minimal system of complex exponentials in \( L^2[-\pi, \pi] \).

\[ \square \]

**Remark 4.1** The author does not know whether the system \( e(\lambda) \) in Theorem 4.1 is always a Riesz sequence or not. But some examples of Riesz sequences for \( L^2[-\pi, \pi] \) seen so far satisfy the condition in Theorem 4.1 as shown by the examples in the next section.

### 5. Examples and remark

The first example is given in [S, Theorem 2.8] as an example of a Riesz sequence of complex exponentials which it cannot be complemented up to a Riesz basis of complex exponentials.

**Example 5.1** Let \( \lambda = \{ \pm(n + \sqrt{n}) \}_{n>1} \) and \( e(\lambda) \equiv \{ e^{\pm i(n + \sqrt{n}) t} \}_{n>1} \).

If we take \( \delta_n = \sqrt{n} \) in Theorem 4.1, then we see that the system \( e(\lambda) \) can be complemented up to a complete and minimal system of complex exponentials in \( L^2[-\pi, \pi] \).

Next we deal with the next example. We may refer to [Y4, p. 136, Theorem 5 and p. 138, Theorem 6].

**Example 5.2** Let \( \lambda = \{ \lambda_n \} \) be a sequence of real numbers such that
\[
\lambda_{n+1} - \lambda_n \geq \gamma > 1 \quad (n = 0, 1, 2, \ldots),
\]
\[
\lambda_{-n} = -\lambda_n \quad (n = 0, 1, 2, \ldots).
\]
Then $e(\lambda) \equiv \{e^{i\lambda_n t}\}$ is a Riesz sequence which has the excess $E(\lambda) = -\infty$ in $L^2[-\pi, \pi]$. Now we can write
\[ \lambda_{n+1} - \lambda_n = 1 + \varepsilon_n, \quad \varepsilon_n \geq \varepsilon > 0 \ (n = 0, 1, 2, \ldots). \]
So we have
\[ \lambda_n = n + \sum_{k=0}^{n-1} \varepsilon_k, \quad n \geq 1. \]
If we take
\[ \delta_n = \sum_{k=0}^{n-1} \varepsilon_k, \quad n \geq 1, \]
there exists a positive integer $n_0$ such that $\delta_{n_0} \geq 1$. We see by Theorem 4.1 that $e(\lambda') \equiv \{e^{i\lambda_n t}\}_{|n| \geq n_0}$ can be complemented up to a complete and minimal system of complex exponentials. Consequently, by [L, p. 7, Theorem 6], the system $e(\lambda)$ can also be complemented up to a complete and minimal system of complex exponentials in $L^2[-\pi, \pi]$.

**Remark 5.1** Let $n^+(r)$ denote the largest number of points from $\lambda$ to be found in an interval of length of $r$ (see [S, p. 133]) and we define
\[ D^+(\lambda) = \lim_{r \to \infty} \frac{n^+(r)}{r}. \]
Then K. Seip has proved in [S, Theorem 2.2] that if $e(\lambda)$ is a Riesz sequence, we have $D^+(\lambda) \leq 1$. Moreover, he has proved in [S, Theorem 2.4] that if $\lambda$ satisfies $D^+(\lambda) < 1$, $e(\lambda)$ can be complemented up to a Riesz basis of complex exponentials in $L^2[-\pi, \pi]$. Consequently, the problem is whether every Riesz sequence $e(\lambda)$ satisfying $D^+(\lambda) = 1$ can be complemented up to a complete and minimal system of complex exponentials in $L^2[-\pi, \pi]$. If, in Theorem 4.1,
\[ \lim_{n \to \infty} \frac{\delta_n}{n} = 0, \]
then we obtain $D^+(\lambda) = 1$.

**Acknowledgment** The author thanks Professor Kristian Seip for his suggestions and kindness. Also the author thanks the referee for his useful comments.
References


