Approximations of hypersingular integral equations by the quadrature method

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Abstract. A numerical method is proposed and investigated for the hypersingular integral equations defined in Banach spaces. The hypersingular integral equations belong to a wider class of singular integral equations having much more stronger singularities.

The proposed approximation method is an extension beyond the quadrature method. Moreover an error estimates theory is introduced for the hypersingular integral equations by proving the proper theorem. Finally, the inequalities valid between the exact solutions of the hypersingular integral equations and the corresponding approximate solutions, are proposed and proved.

Key words: hypersingular integral equations, singularity, quadrature method, error estimate, Banach spaces.

1. Introduction

The hypersingular integral equations consist to a wider class of singular integral equations. In particular the kernel of such integral equations has a stronger singularity as compared to the finite-part singular integral equations. Hence, there is a big interest for the numerical evaluation of the hypersingular integral equations, as closed form solutions are not possible to be determined.

J. Hadamard [1], [2] was the first scientist who introduced the concept of finite-part integrals, and L. Schwartz [3] studied very basic properties of them. Several years later, H.R. Kutt [4] proposed some algorithms for the numerical solution of the finite-part singular integrals and studied the difference between a finite - part integral and a ”generalized principal value integral”.

On the other hand, M.A. Golberg [5] investigated the convergence of several numerical methods for the solution of finite-part integrals. He proposed a method, which was an extension beyond the Galerkin and collocation methods [6]. Also, A.C. Kaya and F. Erdogan [7], [8] introduced and
investigated several problems of elasticity and fracture mechanics, which are reduced to the solution of finite-part singular integral equations.

Moreover, by E.G. Ladopoulos [9]–[15] were proposed several numerical methods for the solution of the finite-part singular integral equations of the first and the second kind. He also applied this type of singular integral equations to the solution of very important problems of elasticity, fracture mechanics and aerodynamics. Beyond the above, E.G. Ladopoulos, V.A. Zisis and D. Kravvaritis [16], [17] used functional analysis for the solution of finite-part singular integral equations. Hence, they studied such type of singular integral equations defined in general Hilbert spaces and $L_p$ spaces and applied them to several crack problems.

In the present research are introduced and investigated the hypersingular integral equations, which have stronger singularity in comparison to the finite-part singular integral equations. Thus, the hypersingular integral equations belong to a wider class of integral equations with kernels of very strong singularities.

A numerical method is proposed for the solution of the hypersingular integral equations, defined in Banach spaces. The proposed approximation method is an extension beyond the quadrature method.

Furthermore, an error estimates theory is proposed for the hypersingular integral equations, by proving the corresponding theorem. Thus the inequalities which are valid between the exact solutions of the hypersingular integral equations and the corresponding approximate solutions, are investigated and proved.

The hypersingular integral equations are used for the solution of several important problems of engineering mechanics, and especially in the theories of elasticity, fracture mechanics, fluid mechanics and aerodynamics.

2. Numerical evaluation methods for hypersingular integral equations

**Definition 2.1** An equation of the following form is called hypersingular integral equation:

$$\int_a^b \frac{u(x)dx}{|x-t|^{\lambda}} = f(t), \quad 1 < \lambda < 3$$

(2.1)
where \( u(x) \) is the unknown function and \( f(t) \) is a known function such as \( f(t) \in C^\infty[a, b] \).

**Theorem 2.1** Let the hypersingular integral equation (2.1) and suppose that following conditions are satisfied:

\[
\begin{align*}
u(x) &= \begin{cases} k_1(x-a)^{(\lambda-1)/2} + k_2(x-a)^{(\lambda+1)/2} + \omega_1(x), & \text{for } x = a \\ k_1'(b-x)(\lambda-1)/2 + k_2'(b-x)(\lambda+1)/2 + \omega_2(x), & \text{for } x = b \end{cases} \\
\end{align*}
\]

(2.2)

where the functions \( \omega_1''(x), \omega_2''(x) \) are Hölder-continuous with exponent \( \varepsilon > 0 \).

Then the hypersingular integral equation (2.1) is approximated by the quadrature formula:

\[
\begin{align*}
R(t_j) &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \frac{u(t_i)}{|x-t_j|^\lambda} \, dx \\
\end{align*}
\]

(2.3)

and its error function \( \Delta(t_j) \) satisfies the estimate:

\[
\begin{align*}
|\Delta(t_j)| &\leq \begin{cases} D \left( \frac{h^{3-\lambda}}{\delta_j^{(5-\lambda)/2}} + \frac{h}{\delta_j^{(\lambda+1)/2}} \right), & \text{for } \lambda \neq 2 \\ Dh \frac{|\ln h|}{\delta_j^{3/2}}, & \text{for } \lambda = 2 \end{cases} \\
\end{align*}
\]

(2.4)

where \( h = (b-a)/n \), \( D \) a constant and \( \delta_j \) the distance of the point \( t_j \) from the boundary of the segment \([a, b] \).

**Proof.** The hypersingular integral in the left hand side of (2.1) is understood in its principal value sense: [5], [15]

\[
\begin{align*}
I(t) &= \int_a^b \frac{u(x)}{|x-t|^\lambda} \, dx \\
&= \lim_{\varepsilon \to 0} \left[ \int_a^{t-\varepsilon} \frac{u(x)}{|x-t|^\lambda} \, dx + \int_{t+\varepsilon}^b \frac{u(x) - u(t)}{|x-t|^\lambda} \, dx + u(t) \int_{t-\varepsilon}^{t+\varepsilon} \frac{dx}{|x-t|^\lambda} + u(t) \int_{t+\varepsilon}^b \frac{dx}{|x-t|^\lambda} + 2u(t) \frac{\varepsilon^{-\lambda+1}}{1-\lambda} \right] \\
\end{align*}
\]

(2.5)
For the numerical evaluation of the integral $I(t)$, then the following points are used:

$$x_i = a + ih, \ i = 0, 1, \ldots, n \quad \text{and} \quad t_j = a + \left(j + \frac{1}{2}\right)h, \ j = 0, 1, \ldots, n - 1$$

where $h = (b - a)/n$ and the quadrature formula (2.3) is applied.

Furthermore consider the point $t_j$ be at the distance $\delta_j$ from the boundary of the segment $[a, b]$, where $\delta_j \geq 5h$.

If $\Delta(t_j)$ is the error function, then it is valid:

$$|\Delta(t_j)| = |I(t_j) - R(t_j)| - \left| \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \frac{u(x) - u(t_i)}{|x - t_j|^{\lambda}} dx \right|$$

$$\leq \left| \sum_{s_i \in [a, b]} \int_{s_i} \frac{u(x) - u(t_i)}{|x - t_j|^{\lambda}} dx \right| + \left| \sum_{s_i \in N_h} \int_{s_i} \frac{u(x) - u(t_i)}{|x - t_j|^{\lambda}} dx \right|$$

$$= \Delta_1 + \Delta_2 \tag{2.6}$$

where the set $N_h$ consists of segments $s_i = [x_i, x_{i+1}]$.

Beyond the above, consider the following equality to be valid:

$$\sum_{s_i \in N_h} \int_{s_i} \frac{u'(t_j)(x - t_i)}{|x - t_j|^{\lambda}} dx$$

$$= \sum_{s_i \in N_h} \int_{s_i} \frac{u'(t_j)(x - t_j)}{|x - t_j|^{\lambda}} dx + \sum_{s_i \in N_h} \int_{s_i} \frac{u'(t_j)(t_j - t_i)}{|x - t_j|^{\lambda}} dx$$

$$= \Gamma_1 + \Gamma_2 \tag{2.7}$$

Since the set $N_h$ is symmetric with respect to $t_j$, then follows: $\Gamma_1 = \Gamma_2 = 0$.

The following formulae hold:

$$u(x) - u(t_i) - u'(t_j)(x - t_i) = u''(z_{ij})(z_i - t_j)(x - t_i) = a_{ij} \tag{2.8}$$

where $x \in [x_i, x_{i+1}]$, $t_i \in [x_i, x_{i+1}]$ and $z_{ij} \in [z_i, t_j]$.

On the other hand, since $u(x)$ satisfies condition (2.2), then the following inequality is valid:

$$|u''(z_{ij})| \leq D\delta_j^{(\lambda-5)/2} \tag{2.9}$$
from which follows:

\[
\Delta_2 = \left| \sum_{s_i \in \mathcal{N}_h} \int_{s_i} \frac{u'(t_j)(x - t_i)}{|x - t_j|^{\lambda}} \, dx + \sum_{s_i \in \mathcal{N}_h, i \neq j} \int_{s_i} \frac{a_{ij}(x)}{|x - t_j|^{\lambda}} \, dx \right|
\]

\[
+ \int_{s_i}^{x_{j+1}} \frac{a_{jj}(x)}{|x - t_j|^{\lambda}} \, dx \right| \leq D \left( \frac{h^{3-\lambda}}{\delta_j^{(5-\lambda)/2}} \int_{t_j+h/2}^{t_j+\delta_j/2} \frac{dx}{|x - t_j|^{\lambda-1}} \right.
\]

\[
+ \frac{1}{\delta_j^{(5-\lambda)/2}} \int_{t_j-h/2}^{t_j+\delta_j/2} \frac{dx}{|x - t_j|^{\lambda-2}} \right) \]

(2.10)

and thus:

\[
\Delta_2 \leq \begin{cases} 
D \left( \frac{h^{3-\lambda}}{\delta_j^{(5-\lambda)/2}} + \frac{h}{\delta_j^{(\lambda+1)/2}} \right), & \text{for } \lambda \neq 2 \\
Dh \left| \frac{\ell \nu h}{\delta_j^{3/2}} \right|, & \text{for } \lambda = 2
\end{cases} \]

(2.11)

By the same way by applying (2.2), then follows inequality:

\[
\Delta_1 \leq \max_s \left\{ \left| \sum_{s_i \in [a, b]} \int_{s_i} \frac{g_s(x) - g_s(t_i)}{|x - t_j|^{\lambda}} \, dx \right| + \left| \sum_{s_i \in [a, b]} \int_{s_i} \frac{d_s(x) - d_s(t_i)}{|x - t_j|^{\lambda}} \, dx \right| \right\}
\]

\[
= \Delta_1' + \Delta_1'', \quad s = 1, 2
\]

(2.12)

where:

\[

g_1(x) = k_1(x - a)^{(\lambda-1)/2} \\
g_2(x) = k_1'(b - x)^{(\lambda-1)/2} \\
d_1(x) = k_2(x - a)^{(\lambda+1)/2} + \omega_1(x) \\
d_2(x) = k_2'(b - x)^{(\lambda+1)/2} + \omega_2(x)
\]

(2.13)

Moreover, following inequality holds for the functions \( g_s(x) \):

\[
|g_s(x) - g_s(t_i)| \leq Dh, \quad \text{for } x \in s_i
\]

(2.14)
and thus:

\[ \Delta_1'' \leq \frac{D_1 h}{\delta_j^{(\lambda+1)/2}} \]  \hspace{1cm} (2.15)

By denoting further by \( \Omega_1 \) the set of segments \( s_i \in [a, b] \) which are on the left from point \( t_j \) and by \( \Omega_2 \) the set of such segments which are on the right from \( t_j \), then follows:

\[ \Delta_1' \leq \left| \sum_{s_i \in \Omega_1} \int_{s_i} \frac{g_s(x) - g_s(t_i)}{|x - t_j|^\lambda} \, dx \right| + \left| \sum_{s_i \in \Omega_2} \int_{s_i} \frac{g_s(x) - g_s(t_i)}{|x - t_j|^\lambda} \, dx \right| = Z_1 + Z_2 \]  \hspace{1cm} (2.16)

On the other hand, by applying the generalized mean-value theorem we have:

\[ Z_1 = h^{(\lambda-1)/2} \sum_{s_i \in \Omega_1} \left[ (i + \xi_i)^{(\lambda-1)/2} - \left( i + \frac{1}{2} \right)^{(\lambda-1)/2} \right] \int_{s_i} \frac{dx}{|x - t_j|^\lambda} \leq \frac{D h^{(\lambda-1)/2}}{m} \sum_{i=0}^{m} \frac{1}{(i + 1)^{(3-\lambda)/2}} \int_{a}^{t_j - \delta_j/2 + h} \frac{dx}{|x - t_j|^\lambda} \]  \hspace{1cm} (2.17)

where \( m \) is the number of segments that belong to \( \Omega_1 \), \( mh > \delta_j/2 - h \) and \( 0 < \xi_i < 1 \).

Thus inequality (2.17) reduces to:

\[ Z_1 \leq \frac{D_1 h}{\delta_j^{(\lambda+1)/2}} \]  \hspace{1cm} (2.18)

By the same way can be proved a similar inequality for \( Z_2 \). Hence, from inequalities (2.15) and (2.18) follows:

\[ \Delta_1 \leq \frac{D_1 h}{\delta_j^{(\lambda-1)/2}} \]  \hspace{1cm} (2.19)

and the estimate (2.4) is proved. \( \square \)

3. Error estimates for hypersingular integral equations

**Theorem 3.1** Consider the hypersingular integral equation (2.1) where \( f(t) \in C^\infty[a, b] \), with an approximate solution \( u_h(t_i) \) given from the system:
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\[ \sum_{i=0}^{n-1} u_h(t_i) \int_{x_i}^{x_{i+1}} \frac{dx}{|x-t_j|^\lambda} = f(t_j), \quad j = 0, 1, \ldots, n-1 \quad (3.1) \]

Then the values \( u(t_k) \) of an exact solution to (2.1) and the values \( u_h(t_k) \) of the approximate solution obtained from (3.1) satisfy the following inequalities:

\[ |u(t_k) - u_h(t_k)| \leq Dh(\lambda-1)/2, \quad 1 < \lambda < 2 \]
\[ |u(t_k) - u_h(t_k)| \leq Dh|\ln h|^2, \quad \lambda = 2 \]
\[ |u(t_k) - u_h(t_k)| \leq Dh(3-\lambda)/2, \quad 2 < \lambda < 3 \]

for \( k = 0, 1, \ldots, n-1 \), where \( h = (b-a)/n \) and \( D \) a constant.

Proof. Let the system of equations:

\[ \sum_{i=0}^{n-1} u_h(t_i)c_{ij} = f(t_j) \]
\[ \sum_{i=0}^{n-1} u(t_i)c_{ij} = f(t_j) + \Delta f, \quad j = 0, 1, \ldots, n-1 \quad (3.3) \]

where \( \Delta f \) is the error of the quadrature formula.

From (3.3) one obtains:

\[ \sum_{i=0}^{n-1} [u(t_i) - u_h(t_i)]d_{ij} = \Delta f(t_j) \quad (3.4) \]

and thus:

\[ \Delta u(t_k) = u(t_k) - u_h(t_k) = \sum_{\ell=0}^{n-1} x_{k\ell} \Delta f(t_{\ell}) \quad (3.5) \]

The general Fourier operator will be further used:

\[ \frac{1}{2\pi} \int_{0}^{2\pi} H(\phi) x_\phi(x_\phi) M_n(y - \phi) d\phi = F(y) \quad (3.6) \]

where:

\[ H(\phi) = \sum_{\ell=-\infty}^{\infty} d_{\ell\phi} e^{i\ell \phi} \]
\[ x(\varphi) = \sum_{\ell=0}^{n-1} u_h(t_\ell) e^{i\ell\varphi} \]

\[ F(y) = \sum_{\ell=0}^{n-1} f(t_\ell) e^{i\ell\varphi} \]  \hspace{1cm} (3.7)

\[ M_n(y) = \sum_{\ell} e^{i\ell y} \]

\[ d_{ij} = \int_{x_i}^{x_{i+1}} \frac{dx}{|x - t_j|^{\lambda}}. \]

Furthermore eqn (3.6) reduces to the following equation:

\[ \frac{1}{2\pi} \int_0^{2\pi} H_n(\varphi) x(\varphi) M_n(y - \varphi) d\varphi = F(y) \]  \hspace{1cm} (3.8)

in which:

\[ H_n(\varphi) = \sum_{\ell=-n}^{n} d_{0\ell} e^{i\ell\varphi} \]  \hspace{1cm} (3.9)

Also in order to be obtained estimates for the inverse matrix, an approximation formula is necessary for the Fourier coefficients \( x^{(n)}_{k\ell} \) of the function \( x^{(\varphi)}_k = \sum_{\ell=0}^{n-1} x_{k\ell}(n)e^{i\ell\varphi} \), where \( x_k \) denotes a solution of the following equation:

\[ \frac{1}{2\pi} \int_0^{2\pi} H_n(\varphi) x_k(\varphi) M_n(y - \varphi) d\varphi = e^{iky}, \]

\[ k = 0, 1, \ldots, n - 1 \]  \hspace{1cm} (3.10)

Let us consider the equation:

\[ \frac{1}{2\pi} \int_0^{2\pi} H_n(\varphi) g_k(\varphi) M_n(y - \varphi) d\varphi = e^{iky} \]  \hspace{1cm} (3.11)

where:

\[ H_n(\varphi) = \sum_{\ell=-n}^{n} a_{\ell} e^{i\ell\varphi} \]

\[ a_{\ell} = -c_{0\ell} h^{\lambda-1} \]  \hspace{1cm} (3.12)
and the solutions $x_k(\varphi)$ and $g_k(\varphi)$ are related by the following formula:

$$x_k(\varphi) = -g_k(\varphi)h^{\lambda-1} \quad (3.13)$$

Moreover in order to study the properties of the function $H_n(\varphi)$ consider the set of segments $[\pi/(16n), \pi/(8n)]$, $n = 1, 2, \ldots$ which forms a covering of the half-open interval $(0, \pi/8]$. By choosing an arbitrary $x \in (0, \pi/8]$, then there exists a minimal number $N$ so that $x \in [\pi/(16N), \pi/(8N)]$.

For $\pi/(16N) \leq \varphi \leq \pi/(8N)$ and $N \leq \ell \leq 2N$ the inequality $\sin^2(\ell\varphi/2) > \delta > 0$ is obtained, from which follows:

$$H_n(\varphi) = -2\sum_{\ell=N}^{2N} a_\ell \sin^2(\ell\varphi/2) \geq 2\delta \sum_{\ell=N}^{2N} |a_\ell| \geq D \varphi^{\lambda-1} \quad (3.14)$$

By choosing $D > 0$ not depending on $N$, then from (3.14) follows that inequality:

$$H(\varphi) > D(\varphi)^{\lambda-1} \quad (3.15)$$

holds for $0 \leq \varphi \leq \pi/8$.

Similarly following inequality:

$$H(\varphi) > D(2\pi - \varphi)^{\lambda-1} \quad (3.16)$$

holds for $15\pi/8 \leq \varphi \leq 2\pi$.

Beyond the above consider $H_n(\varphi) = a_0 + \sum_{\ell=1}^{n} a_\ell \cos(\ell\varphi)$. Because of (3.16), inequality $H_n(\varphi) - H_n(0) > D \varphi^{\lambda-1}$ holds for $\pi/(16n) \leq \varphi \leq \pi/8$.

From the inequality

$$H_n(0) \geq D_1 \sum_{\ell=n+1}^{\infty} \ell^{-\lambda} \geq D_2(n+1)^{1-\lambda}$$

follows that there exists a constant $B_0 > 0$ not depending on $n$ and for which the inequality $H_n(0) \geq B_0 \varphi^{\lambda-1}$ holds for $0 \leq \varphi \leq \pi/(16n)$.

Thus by setting $B_0^* = \min(B_0/2, D)$ one has:

$$H_n(\varphi) \geq \frac{1}{2} H_n(0) + B_0^* \varphi^{\lambda-1} \quad (3.17)$$

for $0 \leq \varphi \leq \pi/8$.

Consider further the linear space $E$ spanned by the functions $e^{i\ell\varphi}$, $\ell =
0, 1, . . . , n − 1 and having the following norms:

\[
\|x\|_1 = \left( \int_0^{2\pi} H_n(\varphi)|x(\varphi)|^2 d\varphi \right)^{1/2}
\]

(3.18)

\[
\|x\|_2 = \max_{\|y(\varphi)\|=1} \left| \int_0^{2\pi} x(\varphi)y(\varphi) d\varphi \right|
\]

By \(E(\| \cdot \|_1)\) and \(E(\| \cdot \|_2)\) we denote the linear spaces \(E\) having the norms \(\| \cdot \|_1\) and \(\| \cdot \|_2\), respectively.

Furthermore the operator \(\Psi(x) = (2\pi)^{-1} \int_0^{2\pi} H_n(\varphi)x(\varphi) B_n(\Psi-\varphi) d\varphi\) maps isometrically the space \(E(\| \cdot \|_1)\) into \(E(\| \cdot \|_2)\).

Thus following inequality holds:

\[
\|e^{ik\varphi}\|_2 \leq D \left( \int_0^{2\pi} \frac{d\varphi}{H_n(\varphi)} \right)^{1/2}
\]

(3.19)

From (3.17) and (3.19) follows:

\[
\|g_k(\varphi)\|_1 = \|e^{ik\varphi}\|_2 \leq \begin{cases} 
D, & \text{for } 1 < \lambda < 2 \\
D \sqrt{|\ell n h|}, & \text{for } \lambda = 2 \\
D / h^{(\lambda-2)/2}, & \text{for } 2 < \lambda < 3
\end{cases}
\]

(3.20)

In order to calculate the Fourier coefficients \(x(\varphi)\) of the functions \(x(\varphi), \|x(\varphi)\|_1 = 1\), one has:

\[
|x_\ell| = \frac{1}{2\pi} \left| \int_0^{2\pi} x(\varphi)e^{-i\ell\varphi} d\varphi \right|
\]

\[
\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{b(\varphi)}{\sqrt{H_n(\varphi)}} d\varphi \leq D \left( \int_0^{2\pi} \frac{d\varphi}{H_n(\varphi)} \right)^{1/2}
\]

(3.21)

and thus \(|x_\ell|\) satisfy estimates which are similar to (3.20).

Hence, the Fourier coefficients \(x_{k\ell}\) of the functions \(x_\varphi(\varphi)\) which are solutions of the problem (3.10), \(x_k(\varphi) = g_k(\varphi)h^{\lambda-1}\) satisfy following inequalities:

\[
|x_{k\ell}| \leq Dh^{\lambda-1}, \quad 1 < \lambda < 2
\]

\[
|x_{k\ell}| \leq Dh|\ell n h|, \quad \lambda = 2
\]

\[
|x_{k\ell}| \leq Dh, \quad 2 < \lambda < 3
\]

(3.22)

As \(x_{k\ell}\) belong to the \(k\)th row of the inverse matrix, then equations
(3.22) denote estimates for the elements of the inverse matrix for (3.1).

Finally from eqs (2.4) and (3.22) we obtain:

\[
|\Delta u(t_k)| \leq \sum_{j=0}^{n-1} Dh^{\lambda-1}\left[ \frac{h^{3-\lambda}}{\delta_j^{(5-\lambda)/2}} + \frac{h}{\delta_j^{(\lambda+1)/2}} \right]
\]

\[
\leq D_1 \left[ \sum_{j=0}^{n/2} h^{(5-\lambda)/2} \frac{h^2}{(j+1)(5-\lambda)/2} + \sum_{j=0}^{n/2} h^{(1+\lambda)/2} \frac{h^{\lambda}}{(j+1)(\lambda+1)/2} \right]
\]

\[
\leq D_2 h^{(\lambda-1)/2} \tag{3.23}
\]

Also, the proof for \( \lambda = 2 \) and \( 2 < \lambda < 3 \) is done by the same way and thus the Theorem is proved. \( \square \)

4. Conclusions

An approximation method has been proposed for the numerical evaluation of the hypersingular integral equations defined in Banach spaces. The hypersingular integral equations are a very special class of singular integral equations having kernels with very strong singularities, as compared to the known finite-part singular integral equations.

The numerical method which was used is an extension beyond the quadrature method. It was therefore proved that the quadrature method is a suitable approximation method for the numerical solution of the hypersingular integral equations. Same method has been successfully used in the past for the numerical evaluation of the non-linear singular integral equations [18]–[28].

Also, an error estimates theory was proposed for the hypersingular integral equations, by proving the suitable theorems. Thus it was shown that same inequalities are valid between the exact solutions of the hypersingular integral equations and the corresponding numerical solutions.

Finally, the hypersingular integral equations are very important for the solution of basic problems of engineering mechanics and mathematical physics, like for example problems of elasticity, fracture mechanics, fluid mechanics and aerodynamics. Hence, there is a big interest for further research on the numerical evaluation of the hypersingular integral equations.
References


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