Forward limit sets of singularities for the Lozi family

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Abstract. The Lozi family is a two-parameter family of piecewise affine uniformly hyperbolic maps on $\mathbb{R}^2$ with strange attractors. We find an open set $\mathcal{O}$ in the parameter space such that, for almost every parameter in $\mathcal{O}$, the forward limit set of a point in the $y$-axis which is a singularity in a trapping region coincides with the strange attractor. This is an extension of the corresponding result about turning orbits in the dynamical core of tent maps on $\mathbb{R}$ by Brucks and Misiurewicz.

Key words: Lozi attractors, singularity set for piecewise hyperbolic maps, $\omega$-limit set.

1. Introduction

The Lozi map is a homeomorphism on $\mathbb{R}^2$ given by

$$f_{a,b}(x, y) = (1 - a|x| + y, bx)$$

for $(x, y) \in \mathbb{R}^2$ where $a$ and $b$ are real parameters. This family was introduced by Lozi [12] as an piecewise affine analogue of the Hénon family, which is now one of the central subjects of study in dynamical system theory [6, 7, 15, 16, 17, 18, 19]. Misiurewicz showed that the map $f_{a,b}$ admits a unique strange attractor $\Lambda_{a,b}$ if $(a, b)$ belongs to the open set $\mathcal{M}$ defined by the inequalities:

$$\begin{cases} 
0 < b < 1, & a > b + 1, \ 2a + b < 4, \\
\frac{a\sqrt{2}}{2} > b + 2, & b < (a^2 - 1)/(2a + 1).
\end{cases}$$

(1)

This strange attractor (the Lozi attractor) has “almost” hyperbolic structure, that is, there is a uniform hyperbolic structure out of the $y$-axis where the Lozi maps are not differentiable, see [13]. However, by the influence of the singularities in the $y$-axis, the dynamics of the Lozi maps are quite delicate [8, 9, 10, 11].

To state our main result, we describe trapping region and singularity set of the Lozi maps, as follows. For any $(a, b) \in \mathcal{M}$, $f_{a,b}$ has a saddle fixed

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point \( p_{a,b} = (1/(1 + a - b) , b/(1 + a - b)) \) which is contained in the first quadrant. The unstable set \( W^u(p_{a,b}) \) of \( p_{a,b} \) contains the line segment that connects \( p_{a,b} \) and the point \( z_{a,b} = ((2 + a + \sqrt{a^2 + 4b})/(2 + 2a - 2b), 0) \) on the \( x \)-axis. The triangle \( T_{a,b} \) with vertices \( z_{a,b}, f_{a,b}(z_{a,b}) \) and \( f_{a,b}^2(z_{a,b}) \) is a trapping region of \( f_{a,b} \), that is, \( f_{a,b}(T_{a,b}) \subset T_{a,b} \). By [13], the Lozi attractor is given by

\[
\Lambda_{a,b} = \cap_{i \geq 0} f_{a,b}^i(T_{a,b}),
\]

which coincides with the closure of \( W^u(p_{a,b}) \). We denote by \( \mathcal{Y}_{a,b} \) the segment of the \( y \)-axis in \( T_{a,b} \). In this paper, we consider the forward orbits of the singularities in \( \mathcal{Y}_{a,b} \), and show that their \( \omega \)-limit sets coincide with \( \Lambda_{a,b} \) for almost every \( (a,b) \) in some parameter region. The main result is

**Main Theorem** There exists an open set \( \mathcal{O} \subset \mathcal{M} \) whose closure contains \( (2,0) \) such that, for Lebesgue almost every \( ((a,b), z) \in \{((a,b), z) \mid (a,b) \in \mathcal{O}, z \in \mathcal{Y}_{a,b} \} \), the \( \omega \)-limit set \( \omega(z, f_{a,b}) \) coincides with the Lozi attractor \( \Lambda_{a,b} \).

When \( b = 0 \), the Lozi maps are equivalent to the tent maps \( t_a(x) = 1 - a|x| \). It has a turning point \( x = 0 \) whose forward orbit can not escape from its dynamical core \( \Lambda(t_a) = [t^2_a(0), t_a(0)] \) for \( 1 < a < 2 \). Brucks and others [3] found a \( G_\delta \)-dense subset of \( a \in [\sqrt{2}, 2] \) such that the forward orbit of \( x = 0 \) is dense in \( \Lambda(t_a) \). Brucks and Misiurewicz showed in [4] that almost every \( a \in [\sqrt{2}, 2] \) satisfies \( \omega(0, t_a) = \Lambda(t_a) \). The main theorem of this paper is an extension of this result to the 2-dimensional context. In the 1-dimensional case, Brucks and Buczolich [1] showed that the complement of such parameters is \( \sigma \)-porous, and Bruin [5] showed that, for almost every parameter value, the turning orbit is typical for an absolutely continuous invariant probability measure. (See also [2].) However, the corresponding results for the Lozi family are not known.

2. **Proof of Main Theorem**

Let \( f_{a,b} \) be the Lozi family, and \( \mathcal{M} \) the parameter set defined by (1). Let \( \mathcal{O} \subset \mathcal{M} \) be a small open set which is specified concretely in the following sections. At this stage, it is enough to keep in mind that it is small and \( (2,0) \in \text{cl}(\mathcal{O}) \), where \( \text{cl}(\cdot) \) is the closure of the corresponding set. Let us fix a point \( (a_0, b_0) \in \mathcal{O} \) arbitrarily in the argument belows. Let \( I \subset \mathbb{R} \) be a
neighborhood of \( a_0 \) such that \( I \times \{ b_0 \} \subset \mathcal{O} \). For any \( a \in I \), we abbreviate \( f_{a,b_0} \) to \( f_a \). For each \( a \in I \), \( f_a \) has a saddle fixed point

\[
p_a = \left( \frac{1}{1 + a - b_0}, \frac{b_0}{1 + a - b_0} \right).
\]

The stable and unstable set of \( p_a \) are denoted by \( W^s(p_a) \) and \( W^u(p_a) \), respectively. As illustrated in the Fig. 1, \( W^s(p_a) \) contains the line segment \( S_a \) connecting \( p_a \) and the point

\[
w_a = \left( 0, \frac{2b_0 - a - \sqrt{a^2 + 4b_0}}{2(1 + a - b_0)} \right)
\]
on the y-axis. Also, \( W^u(p_a) \) contains the line segment \( \mathcal{U}_a \) that connects \( p_a \) and the point

\[
z_a = \left( \frac{2 + a + \sqrt{a^2 + 4b_0}}{2(1 + a - b_0)}, 0 \right)
\]
on the x-axis. Since the expanding eigenvalue of \( (Df_a)p_a \) is negative, we get

\[
W^u(p_a) = \bigcup_{i \geq 0} f_a^i(\mathcal{U}_a).
\]

Since \( (a, b_0) \in \mathcal{O} \), we can check that

\[
(w_a)_y < (f_a^2(z_a))_y, \quad (f_a^{-1}(w_a))_y > (f_a(z_a))_y
\]

where \( (\cdot)_y \) is the y-coordinate of the corresponding point. Therefore, \( S_a \) and \( f_a^2(\mathcal{U}_a) \) intersect transversely, and \( f_a^{-1}(S_a) \) and \( f_a(\mathcal{U}_a) \) intersect transversely for every \( a \in I \), as in Fig. 1. The triangle \( T_a \) with vertexes \( z_a, f_a(z_a) \) and \( f_a^2(z_a) \) is a trapping region. The Lozi attractor is given by

\[
\Lambda_a = \bigcap_{i \geq 0} f_a^i(T_a).
\]

Let us fix a point \( z \in \mathcal{Y}_a \) where \( \mathcal{Y}_a = \{y\text{-axis}\} \cap T_a \). For \( a \in I \) and \( i \geq 0 \), we put

\[
\varphi_i(a) := f_a^i(z).
\]

Set \( \bar{\mathcal{U}} = \bigcup_{a \in I} \mathcal{U}_a \), and consider its cover \( \mathcal{H} \) which consists of all open balls whose radii and central coordinates are both rational, and whose intersec-
tion with $\tilde{U}$ is non-empty. See Fig. 2. For $H \in \mathcal{H}$, we define

$$I_H = \{ a \in I \mid \mathcal{U}_a \cap H \neq \emptyset \},$$

and

$$A_H = \{ a \in I_H \mid \varphi_i(a) \notin H \quad \text{for} \quad \forall i \geq 0 \}.$$

The next lemma is essential in the proof of the main theorem. We denote by $\mu$ the 1-dimensional Lebesgue measure on $I$. 
Lemma 1  For any $H \in \mathcal{H}$, $\mu(A_H) = 0$.

The proof of Lemma 1 is given based on the following claims: for almost every $a \in I_H$ and every neighborhood $U$ of the point $a$, there exist integer $\nu > 0$ and closed interval $J \subset U$ including the point $a$ such that
(A) $\varphi_\nu(J)$ intersects with $f_{\alpha}^{-1}(S_\alpha)$, one of the endpoints of $\varphi_\nu(J)$ belongs to the y-axis, and $1/2 < \text{Length}(\varphi_\nu(J)) < 4$, as in Fig. 3, (see Theorem 6), where Length$(J)$ is the length of $J$;
(B) for any $a_1, a_2 \in J$,
\[
\frac{|d\varphi_\nu(a_1)/da|}{|d\varphi_\nu(a_2)/da|} < 2,
\]
(see Proposition 4),
whose proofs will be presented in the following sections.

Proof of Lemma 1.  Suppose that there exists an open set $H \in \mathcal{H}$ such that $\mu(A_H) > 0$. Take a Lebesgue density point $\alpha$ of $A_H$. By the inclination lemma [14] and the piecewise hyperbolic structure of Lozi maps, for any line segment $l \subset H$ intersecting with the unstable segments $U_\alpha$ transversally, there exists an integer $k > 0$ such that $f_{\alpha}^{-k}(l)$ becomes a V-shaped segment which is piecewise $C^1$-close to $f_{\alpha}^{-1}(S_\alpha)$, as shown in Fig. 3.

Since $f_{\alpha}^{-k}(l)$ is compact, and $H$ is an open set, there is a $c > 0$ such that
\[
N_{2c}(f_{\alpha}^{-k}(l)) \subset f_{\alpha}^{-k}(H),
\]
where $N_{2c}(f_{\alpha}^{-k}(l))$ is a $2c$-neighborhood of $f_{\alpha}^{-k}(l)$. If a neighborhood $U$ of $\alpha$ is sufficiently small, then for any $a \in U$
\[
N_c(f_{\alpha}^{-k}(l)) \subset f_{a}^{-k}(H).  \tag{2}
\]
By claim (A) above, there exists a segment $L \subset J \subset U$ such that
\[
\varphi_\nu(L) \subset \varphi_\nu(J) \cap \bigcap_{a \in J} f_{a}^{-k}(H) \neq \emptyset.
\]
From (2),
\[
\text{Length}(\varphi_\nu(L)) > c > 0,  \tag{3}
\]
where Length$(\cdot)$ is the length of a given arc.
By claim (B) above, we have

\[
\frac{\text{Length}(\varphi_\nu(L))/\mu(L)}{\text{Length}(\varphi_\nu(J))/\mu(J)} = \frac{|\tau_\nu(a_1)|}{|\tau_\nu(a_2)|} < 2,
\]

where \(\tau_\nu(a_i) = d\varphi_\nu(a_i)/da\). For every \(a \in L\), we have \(\varphi_\nu(a) \in \varphi_\nu(L)\), and \(\varphi_{\nu+k}(a) \in H\). Therefore, such a parameter value \(a\) is not contained in \(A_H\).

Thus,

\[
\frac{\mu(L)}{\mu(J)} < \frac{\mu(J \setminus A_H)}{\mu(J)} = 1 - \frac{\mu(J \cap A_H)}{\mu(J)},
\]

and hence

\[
\frac{\text{Length}(\varphi_\nu(L))}{2 \text{Length}(\varphi_\nu(J))} < 1 - \frac{\mu(J \cap A_H)}{\mu(J)}.
\]

Since the diameter of the trapping region is smaller than 4, we have \(\text{Length}(\varphi_\nu(J)) < 4\). Therefore, using (3), we obtain

\[
\frac{\mu(J \cap A_H)}{\mu(J)} < 1 - \frac{\text{Length}(\varphi_\nu(L))}{8} < 1 - \frac{c}{8}.
\]
However, since $\alpha$ is a Lebesgue density point of $A_H$, we have
\[
\frac{\mu(J \cap A_H)}{\mu(J)} > 1 - \frac{c}{8}
\]
for every interval $J \subset U$, if $U$ is sufficiently small. This is a contradiction. \qed

**Proof of Main Theorem.** For any $H \in \mathcal{H}$, from the above Lemma 1, we have $\mu(A_H) = 0$. Since $\mathcal{H}$ is countable,
\[
\mu\left(\bigcup_{H \in \mathcal{H}} A_H\right) \leq \sum_{H \in \mathcal{H}} \mu(A_H) = 0.
\]
That is, for almost every $a \in I_H$, there exists $i \geq 0$ such that $\varphi_i(a) = \varphi_i^t(z) \in H$ where $z \in \mathcal{V}_a$. Since this holds for each element of $\mathcal{H}$, we get
\[
\mathcal{U}_a \subset \omega(z, f_a).
\]
Thus, since $W^u(p_a) = \bigcup_{i \geq 0} f_a^i(\mathcal{U}_a)$, we obtain
\[
\text{cl}(W^u(p_a)) \subset \omega(z, f_a).
\]
Remember that, at the beginning of this section, $(a_0, b_0)$ is an arbitrary point in $\mathcal{O}$. For almost every point $(a, b_0)$ of the horizontal parameter segment in $\mathcal{O}$, the above claim is true. Hence, the main theorem is proved. \qed

3. **Estimations of parameter dependence**

In this section, we first define the open set $\mathcal{O} \subset \mathcal{M}$ of parameters in the main theorem. After that we set an open interval $I$ and a constant $b_0$ such that $I \times \{b_0\} \subset \mathcal{O}$. The goal of this section is to show the Proposition 4 which is used in the proof of Lemma 1.

To begin with, we assume that $\mathcal{O}$ satisfies $(2, 0) \in \text{cl}(\mathcal{O})$ and it is sufficiently small such that, for any $(a, b) \in \mathcal{O},$
\[
f^i_{a, b}(\mathcal{V}_{a, b}) \cap C = \emptyset, \quad 1 \leq i \leq 10; \tag{4}
\]
\[
\sup\{|x| : (x, y) \in T_{a, b}\} < 1.05; \tag{5}
\]
\[
1.9 < \lambda_{a, b} < 2,
\]
where $C = \{(x, y) \in \mathbb{R}^2 : |x| < 1/2\}$ and $\lambda_{a, b} = (a + \sqrt{a^2 - 4b})/2$. Let us define $\tilde{\lambda}_{a, b} = (a - \sqrt{a^2 - 4b})/2$ which satisfies $0 < \tilde{\lambda}_{a, b} < 1 < \lambda_{a, b}$. If a
point \( x = (x, y) \) is not contained in the \( y \)-axis, by [13], each cone

\[
C^u = \{(x, y) \in T_x \mathbb{R}^2 : |y| \leq \tilde{\lambda}_{a, b}|x|\},
\]

\[
C^s = \{(x, y) \in T_x \mathbb{R}^2 : |y| \geq \lambda_{a, b}|x|\},
\]
is invariant by \((Df_{a, b})_x\) and \((Df_{a, b})^{-1}_x\), respectively, and it holds

\[
|(Df_{a, b})_x u| \geq \lambda_{a, b}|u|, \quad |(Df_{a, b})^{-1}_x s| \leq \tilde{\lambda}_{a, b}|s|
\]
for any \( u \in C^u \) and \( s \in C^s \).

We abbreviate \( f_{a, b_0} = f_a \) as \( b_0 > 0 \) is fixed small. For a fixed \( z \in Y_{a, b_0} \) and each \( i \geq 0 \), we put

\[
\varphi_i(a) = f^i_a(z)
\]
and

\[
\tau_i = \tau_i(a) = \frac{d\varphi_i(a)}{da}.
\]

If the Jacobian \((Df_a)_j, 0 \leq j \leq i\), make sense, we have

\[
\begin{cases}
\tau_1 = (0, 0) \\
\tau_{j+1} = (Df_a)_j \tau_j + \eta_{j+1} \quad \text{for } 0 \leq j \leq i
\end{cases}
\]

(6)

where \( \eta_{j+1} = (-|x_j|, 0) \) and \( x_j \) is the \( x \)-coordinate of \( f^i_a(z) \). We say that \( \tau_i(a) \) is well-defined if \( \tau_j(a) \) is given by (6) for all \( 1 \leq j \leq i \).

To estimate \( \tau_i \), let us introduce a pair of reference vectors \((u_i, s_i)\) as follows. Let \( i_0 \geq 2 \) be an integer such that \((x_i, y_i) \notin y\)-axis for each \( 2 \leq i \leq i_0 \), that is, \( \tau_i \) is well-defined for all \( 2 \leq i \leq i_0 \). As shown in the Fig. 4, we first define

\[
u_2 = (-1.05, 0), \quad s_2 = \frac{|u_2|}{|e_2|}e_2,
\]

where

\[
e_2 = (Df^{i_0-2}_a)^{-1}_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in C^s.
\]

For each \( 2 \leq i \leq i_0 \), we define

\[
u_{i+1} = (Df_a)_i u_i, \quad s_{i+1} = \frac{|u_{i+1}|}{|(Df_a)_i s_i|}(Df_a)_i s_i.
\]
So, these reference vectors satisfy
\begin{itemize}
  \item $u_i \in C^u$, $s_i \in C^s$ and $|u_i| = |s_i|$;
  \item $|u_{i+1}| \geq \lambda_{a,b}|u_i|$ and $|s_{i+1}| \geq \lambda_{a,b}|s_i|$.
\end{itemize}

Since $\eta_i = (\pm |x_{i-1}|, 0)$ and (5), we have
\[ |u_i| \geq 1.05 > |\eta_i|, \]
for each $3 \leq i \leq i_0$. Then, there exist $\xi_i, \tilde{\xi}_i \in \mathbb{R}$ such that
\[ \eta_i = \xi_i u_i + \tilde{\xi}_i s_i. \]

Since the slope of the central line of the cones $C^u$ and $C^s$ tend to 0 and 2 as $(a, b) \to (2, 0)$ respectively, if the open set $\mathcal{O}$ is sufficiently small, then we have
\[ 1.1|\eta_i| > |\xi_i u_i| > |\tilde{\xi}_i s_i| \]
for each $3 \leq i \leq i_0$. We get
\[ 1.1|u_2| \geq 1.1|\eta_i| > |\xi_i u_i| = |\xi_i||(Df_a^{i-2})_2 u_2| > \lambda_{a,b}^{i-2}|\xi_i||u_2|. \]
Therefore,
\[ |\tilde{\xi}_i| < |\xi_i| < \frac{1.1}{\lambda_{a,b}^{i-2}}. \]
(7)

We can confirm that $\xi_3 > |\tilde{\xi}_3|$ and $\xi_4 > |\tilde{\xi}_4|$. Using the above decompositions
by reference vectors, provided $\tau_i$ is well-defined, we obtain
\[
\tau_i = (\xi_2 + \xi_3 + \xi_4 + \cdots + \xi_i)u_i \\
+ \left( \frac{|(Df_a)_{i-1}|s_{i-1}|}{|u_i|} \cdot \xi_3 \right) u_i \\
+ \left( \frac{|(Df_a)_{i-1}^2|s_{i-1}|}{|u_i|} \cdot \xi_4 \right) u_i \\
+ \cdots \\
+ \left( \frac{|(Df_a)_{i-1}|s_{i-1}|}{|u_i|} \cdot \xi_{i-1} \right) u_i \\
+ \xi_i) s_i.
\]

We moreover assume that $\mathcal{O}$ is so small that, for any $(a, b) \in \mathcal{O}$,
\[
f_{a, b}(\mathcal{Y}_{a, b}) \subset (0.92, \infty) \times \{0\}.
\]

Hence,
\[
|\xi_2| = \left| \frac{\eta_2}{u_2} \right| = \frac{|x_1|}{|u_2|} > \frac{0.92}{1.05} > 0.87. \tag{8}
\]

By $\tilde{\lambda}_{a, b} \searrow 0$ and $\lambda_{a, b} \nearrow 2$ as $(a, b) \to (2, 0)$, if the open set $\mathcal{O}$ is sufficiently small, then the following condition can be held:
\[
\frac{\xi_2 + (\xi_3 - |\tilde{\xi}_3|) + (\xi_4 - |\tilde{\xi}_4|) - \Gamma(a, b)}{\xi_2 + \xi_3 + |\tilde{\xi}_3| + \xi_4 + |\tilde{\xi}_4| + \Gamma(a, b)} \cdot \lambda_{a, b} > 1.18 \tag{9}
\]

where
\[
\Gamma(a, b) = \frac{1.1(1 + 2\tilde{\lambda}_{a, b})}{\lambda_{a, b}^2(\lambda_{a, b} - 1.1)}.
\]

**Lemma 2** If we take sufficiently small $\mathcal{O}$ with $(2, 0) \in \text{cl}(\mathcal{O}) \subset \mathcal{M}$, there is an integer $i_0 > 2$ such that
- if $\tau_i(a)$ is well-defined for $a \in I$ and $i \geq 2$, then
  \[
  0.1\lambda_{a}^{i-2} < |\tau_i(a)| < 4(1 + \sqrt{2})^{i-1}; \tag{10}
  \]
- if $\tau_i(a)$ is well-defined for $a \in I$ and $i \geq i_0$, then
  \[
  0.1\lambda_{a}^{i-2} < \sqrt{2}|\Pi_x(\tau_i(a))| \tag{11}
  \]

where $\Pi_x$ is a canonical projection from $\mathbb{R}^2$ to the $x$-axis.
Proof. From the above expression of $\tau_i$ by reference vectors, we get

$$|\tau_i| \geq \left\{ \xi_2 + (\xi_3 - |\tilde{\xi}_3|) + (\xi_4 - |\tilde{\xi}_4|) - \sum_{n=5}^{i} (|\xi_n| + |\tilde{\xi}_n|) \right\} |u_i|.$$ 

By (7), we have

$$\sum_{n=5}^{i} (|\xi_n| + |\tilde{\xi}_n|) \leq \frac{2 \cdot (1.1/\lambda_a^3)}{1 - (1.1/\lambda_a)} \frac{2.2}{\lambda_a^2(\lambda_a - 1.1)} < \frac{2.2}{1.9^2(1.9 - 1.1)} < 0.77,$$

Hence, by (8),

$$|\tau_i| \geq \left\{ 0.87 + (\xi_3 - |\tilde{\xi}_3|) + (\xi_4 - |\tilde{\xi}_4|) - 0.77 \right\} |u_i|$$

$$> 0.1\lambda_a^{i-2} |u_2|$$

$$> 0.1\lambda_a^{i-2}.$$ 

The second inequality of (10) is obtained as follows. Since

$$\| (Df_a)_{i} \| = \frac{a + \sqrt{a^2 + 4b}}{2} < 1 + \sqrt{2},$$

we get, for every $i \geq 2$,

$$|\tau_i| \leq \| (Df_a)_{i-1} \| |\tau_{i-1}| + |\eta_i| < (1 + \sqrt{2}) |\tau_{i-1}| + 4.$$ 

Then,

$$|\tau_i| \leq (1 + \sqrt{2})^{i-2} |\tau_2| + 4 \{ (1 + \sqrt{2})^{i-3} + \cdots + 1 \}.$$ 

Since $|\tau_2| = |\eta_2| < 4$,

$$|\tau_i| \leq 4 \{ (1 + \sqrt{2})^{i-2} + (1 + \sqrt{2})^{i-3} + \cdots + 1 \} < 4(1 + \sqrt{2})^{i-1}.$$ 

Hence, (10) is obtained.

Denote that $\tilde{C}^{u+} = \{ (x, y) \in T_x\mathbb{R}^2 : |y| \leq x \}$, $\tilde{C}^{u-} = \{ (x, y) \in T_x\mathbb{R}^2 : |y| \leq -x \}$ and $\tilde{C}^{u} = \tilde{C}^{u+} \cup \tilde{C}^{u-}$. Note that $(Df_a)_x v \in \text{Int}(\tilde{C}^u)$ for any nonzero $v \in \tilde{C}^u$ if $(a, b)$ is close to $(2, 0)$. Since $\eta_{i+1} = ( -|x_i|, 0)$ and $|x_i| < 2$ for any $i > 0$, there exists a constant $u_0 > 0$ such that for any $u \in \tilde{C}^u$ with $|u| \geq u_0$,

$$(Df_a)_x u + \begin{pmatrix} -2 \\ 0 \end{pmatrix} \in \text{Int}(\tilde{C}^u).$$
By (10), if \((a, b)\) is close to \((2, 0)\), then there exists an integer \(i_0 > 2\) such that
- \(\tau_2, \ldots, \tau_{i_0-1} \in \mathcal{C}^{u-} \) and \(\tau_{i_0} \in \mathcal{C}^{u+} \);
- \(|\tau_i| > u_0\) for every \(i \geq i_0\), as \(\tau_i\) is well-defined.

This implies that the norm of \(\Pi_x(\tau_i(a))\) is greater than \(|\tau_i(a)|/\sqrt{2}\) for any \(i \geq 0\). Therefore, the proof is now complete. \(\square\)

If \(\tau_i(a)\) is well-defined, then one can get
\[
\tau_i'(a) = \frac{d\tau_i(a)}{da} = \frac{d^2\varphi_i(a)}{da^2} = \left( \frac{d^2x_i}{da^2}, \frac{d^2y_i}{da^2} \right).
\]

By direct calculations, we have
\[
\frac{d^2x_{i+1}}{da^2} = -\text{sgn}(x_i) \cdot a \cdot \frac{d^2x_i}{da^2} + \frac{d^2y_i}{da^2} - 2 \cdot \text{sgn}(x_i) \cdot \frac{dx_i}{da}
\]
\[
\frac{d^2y_{i+1}}{da^2} = b \cdot \frac{d^2x_i}{da^2},
\]
that is,
\[
\tau_{i+1}'(a) = (Df_a)_{i} \tau_i'(a) + 2 \left( -\text{sgn}(x_i)(dx_i/da) \right).
\]

**Lemma 3** If \(\tau_i(a)\) is well-defined for \(a \in I\), then
\[
|\tau_j'(a)| < 8j(1 + \sqrt{2})^j
\]
for \(j = 1, \ldots, i\).

**Proof.** We prove it by induction. Since \(|\tau_i'(a)| = 0\), the claim holds for \(j = 1\). Suppose that it holds for \(1 \leq j < i\). Using \(|(Df_a)_j| < 1 + \sqrt{2}\) and Lemma 2, we have
\[
|\tau_{j+1}'(a)| \leq \|(Df_a)_j||\tau_j'(a)| + 2 \left| \frac{dx_j}{da} \right|
\]
\[
< (1 + \sqrt{2}) \cdot 8n(1 + \sqrt{2})^j + 2\tau_j
\]
\[
< 8(j + 1)(1 + \sqrt{2})^{j+1}
\]
Then the claim holds for \(j + 1\). Therefore, the lemma is true for each \(j = 1, \ldots, i\). \(\square\)
Proposition 4. For any $\gamma > 0$ there exists $i_1 \geq 1$ such that if $\tau_i$ is well-defined on a closed interval $J \subset I$ for $i \geq i_1$, then

$$\frac{|\tau_i(a_1)|}{|\tau_i(a_2)|} \leq 1 + \gamma$$

for any $a_1, a_2 \in J$.

Proof. When $a_1 = a_2$, the lemma is trivial. Let $a_1, a_2 \in J$ with $a_1 \neq a_2$. Using Lemma 3, we have

$$\frac{|\tau_i(a_1)| - |\tau_i(a_2)|}{|a_1 - a_2|} \leq \sup_{a \in J} |\tau'_i(a)| < 8i(1 + \sqrt{2})^i.$$ 

If $\tau_i$ is well-defined on $J$ for $i \geq i_0$, by Lemma 2 (11),

$$\frac{4}{|a_1 - a_2|} \geq \frac{\left|\Pi_x(\varphi_i(a_1))\right| - \left|\Pi_x(\varphi_i(a_2))\right|}{|a_1 - a_2|} \geq \inf_{a \in J} \left|\Pi_x(\tau_i(a))\right| > \frac{0.1\lambda_{\bar{a}}^{i-2}}{\sqrt{2}},$$

for some $\bar{a} \in J$. Then, we have

$$|\tau_i(a_1)| - |\tau_i(a_2)| < 8i(1 + \sqrt{2})^i|a_1 - a_2| < \frac{32\sqrt{2}i(1 + \sqrt{2})^i}{0.1\lambda_{\bar{a}}^{i-2}}.$$ 

Note that $\lambda_{\bar{a}}^2 > (7/5)(1 + \sqrt{2})$ for any $a \in J$. Then, using Lemma 2 (10), we get

$$\frac{|\tau_i(a_1)|}{|\tau_i(a_2)|} - 1 < \frac{32\sqrt{2}i(1 + \sqrt{2})^i}{0.1\lambda_{\bar{a}}^{i-2}} \frac{1}{0.1\lambda_{a_2}^{i-2}} < 6400(1 + \sqrt{2})^i\left(\frac{5}{7}\right)^{i-2}.$$ 

So, for any $\gamma > 0$, we take an integer $i_1 \geq i_0$ such that, for any $i \geq i_1$,

$$6400(1 + \sqrt{2})^i\left(\frac{5}{7}\right)^{i-2} \leq \gamma.$$ 

□

Lemma 5. There exists an integer $i_2 > 0$ and $\zeta > 1.15$ such that, for any $a \in I$,

$$\frac{|\tau_{i+1}(a)|}{|\tau_i(a)|} > \zeta$$

if $\tau_{i+1}$ is well-defined for given $i \geq i_2$. 
Proof. By (6),
\[ |\tau_{i+1}| \geq |(Df_a)i\tau_i| - |\eta_{i+1}| > |(Df_a)i\tau_i| - 4. \]

Then, using Lemma 2, we get
\[ \frac{|\tau_{i+1}|}{|\tau_i|} > \frac{|(Df_a)i\tau_i|}{|\tau_i|} - \frac{4}{|\tau_i|} > \frac{|(Df_a)i\tau_i|}{|\tau_i|} - \frac{4}{0.1\lambda_{-2}^i}. \]

If
\[ \frac{|(Df_a)i\tau_i|}{|\tau_i|} > 1.18, \tag{12} \]

then we can get an integer \( i_2 > 0 \) such that, for \( i \geq i_2 \),
\[ \frac{|(Df_a)i\tau_i|}{|\tau_i|} - \frac{40}{\lambda_{-2}^i} > 1.18 - \frac{40}{\lambda_{-2}^i} > 1.15. \]

Let us show that (12) is true as follow. By the linear decomposition of \( \tau_i \), we get
\[ |\tau_i| \leq \left\{ \xi_2 + \xi_3 + |\tilde{\xi}_3| + \xi_4 + |\tilde{\xi}_4| + \sum_{n=5}^{i} (|\xi_n| + |\tilde{\xi}_n|) \right\}|u_i| \]
\[ \leq \left\{ \xi_2 + \xi_3 + |\tilde{\xi}_3| + \xi_4 + |\tilde{\xi}_4| + \frac{1.1(1 + 2\lambda_a)}{\lambda_a^2(\lambda_a - 1.1)} \right\}|u_i|, \]

and
\[ |(Df_a)i\tau_i| \geq \left\{ \xi_2 + (\xi_3 - |\tilde{\xi}_3|) + (\xi_4 - |\tilde{\xi}_4|) - \sum_{n=5}^{i} (|\xi_n| + |\tilde{\xi}_n|) \right\}|u_{i+1}| \]
\[ \geq \left\{ \xi_2 + (\xi_3 - |\tilde{\xi}_3|) + (\xi_4 - |\tilde{\xi}_4|) - \frac{1.1(1 + 2\lambda_a)}{\lambda_a^2(\lambda_a - 1.1)} \right\}\lambda_a|u_i|. \]

Then, by (9), we get
\[ \frac{|(Df_a)i\tau_i|}{|\tau_i|} \geq \frac{\xi_2 + (\xi_3 - |\tilde{\xi}_3|) + (\xi_4 - |\tilde{\xi}_4|) - \frac{1.1(1 + 2\lambda_a)}{\lambda_a^2(\lambda_a - 1.1)}}{\xi_2 + \xi_3 + |\tilde{\xi}_3| + \xi_4 + |\tilde{\xi}_4| + \frac{1.1(1 + 2\lambda_a)}{\lambda_a^2(\lambda_a - 1.1)}} \cdot \lambda_a > 1.18. \]

This completes the proof. \( \square \)
4. Usefulness and maturity of parameter arcs

The concepts of usefulness and maturity for parameter intervals of tent maps are introduced in [4]. Let us extend these concepts to the Lozi family.

For \( k \geq 1 \), the parameter interval \( I \) is called \( k \)-useful if

- \( \tau_k \) is well-defined on \( I \),
- there exists \( a_0 \in \partial I \) such that \( \varphi_k(a_0) \in \{ y \text{-axis} \} \),

where \( \partial I \) is the set of endpoints of \( I \). If there exist several \( k \)'s for which \( I \) is \( k \)-useful, we call the largest one order of \( I \) which is denoted by \( \text{Ord}(I) \), whose finitude is ensured by Lemma 2.

Next, we extend the concept of maturity to a subinterval of a useful \( I \) of order \( N \). Let \( \tilde{I} \subset I \) be an open interval. We say that \( \tilde{I} \) is \( k \)-mature if

- there exists some \( k \geq N \) such that \( \tilde{I} \) is \( k \)-useful,
- there exist \( \tilde{a} \in \tilde{I} \) and \( (0, \tilde{y}) \in \{ y \text{-axis} \} \) such that

\[
\varphi_k(\tilde{a}) = f^k_{\tilde{a}}(0, y) = f^{m}_{\tilde{a}}(0, \tilde{y})
\]

for some \( m \in \{1, 2, \ldots, 9\} \).

A point of \( I \) which does not belong to any mature subset of \( I \) is called bad, and a set of all bad points of \( I \) is denoted by \( \mathcal{B} \).

We define partitions of \( I \) inductively. Let \( k > 0 \) be an integer such that \( \varphi_k(I) \cap \{ y \text{-axis} \} = \emptyset \) where \( \varphi_k(I) = \{ \varphi_k(a) : a \in I \} \). Using Lemma 2, we have the smallest integer \( h > 0 \) such that \( \varphi_{k+h}(I) \) intersects transversely at one point of the \( y \)-axis. So, by this intersection, \( \varphi_{k+h}(I) \) is divided into two adjacent segments which are images of two adjacent \( (k+h) \)-useful open intervals of \( I \), respectively, denoted by \( J_1 \) and \( J_2 \). We now get the first partition \( \mathcal{P}_1 = \{ J_1, J_2 \} \) of \( I \). If \( J_i \in \mathcal{P}_1 \) is mature, we set \( \rho(J_i) = \{ J_i \} \); otherwise, by similar steps, we can divide \( J_i \) into two \( (k+h') \)-useful, \( h' > h \), arcs \( J_{i1} \) and \( J_{i2} \), and set \( \rho(J_i) = \{ J_{i1}, J_{i2} \} \). Then we get the second partition \( \mathcal{P}_2 = \bigcup_{J_i \in \mathcal{P}_1} \rho(J_i) \). Similarly, for every \( n \geq 3 \), we obtain the partition \( \mathcal{P}_n = \bigcup_{J \in \mathcal{P}_{n-1}} \rho(J) \) of \( I \).

We claim the following:

**Theorem 6** For almost every \( a \in I \), there is a \( k \)-mature \( \tilde{I} \subset I \) with \( a \in \tilde{I} \) such that

\[
|\langle \varphi_k(\tilde{a}) \rangle_x | \geq \frac{1}{2}
\]

for some \( \tilde{a} \in \partial \tilde{I} \), where \( \langle . \rangle_x \) is the \( x \)-coordinate of the corresponding point. That is, one of the endpoints of \( \varphi_k(\text{cl}(\tilde{I})) \) keeps away from the \( y \)-axis at
least by 1/2.

We will deduce this theorem from Proposition 8 and Proposition 9 stated later. To prove these propositions, we prepare a lemma. Let us define the function $\psi_n$ on $\mathcal{B}$ by

$$\psi_n(a) = \int_{B \cap J} |\tau_k| \, d\mu$$

where $J$ is an element of $\mathcal{P}_n$ that contains $a$ and $k$ is the order of $J$. We set

$$N = \max\{i_1, i_2\},$$

where $i_1$ and $i_2$ are given in Proposition 4 and Lemma 5, respectively. Remember that the constants $\gamma > 0$ and $\zeta > 1.15$ are also presented in Proposition 4 and Lemma 5, respectively.

**Lemma 7**  For $n \geq N$, let $J \in \mathcal{P}_n$ be a $k$-useful interval. If $\mu(B \cap J) > 0$, then

$$\int_{B \cap J} \psi_n+1 \, d\mu \geq \sigma \int_{B \cap J} \psi_n \, d\mu$$

where

$$\sigma = \min \left\{ \zeta, \frac{\zeta^{10}}{2(2 + \gamma)} \right\}.$$

**Proof.** From $\mu(B \cap J) > 0$, $J$ is not mature. Then, there is the smallest integer $m \geq 0$ such that $\xi_{k+m}(J)$ intersects $Y$. Thus, we get the first partition $\rho(J) = \{J_1, J_2\}$. Obviously, $\text{Ord}(J_i) \geq k + m$. Without loss of generality, we may assume that

$$\int_{B \cap J_1} |\tau_{k+m}| \, d\mu \geq \int_{B \cap J_2} |\tau_{k+m}| \, d\mu. \quad (13)$$

By Lemma 5, we have

$$\psi_{n+1}(a) = \int_{B \cap J_i} |\tau_{\text{Ord}(J_i)}| \, d\mu \geq \int_{B \cap J_i} |\tau_{k+m}| \, d\mu \quad (14)$$

for any $a \in J$ and $i = 1, 2$. Also from Lemma 5, we have

$$|\tau_{k+m}| > \zeta^m |\tau_k| > \zeta |\tau_k|. \quad (15)$$

Since $\mu(B \cap J) > 0$, it is impossible that both $J_1$ and $J_2$ are mature. First, we suppose that $B \cap J_2 = \emptyset$, that is, $B \cap J = B \cap J_1$. Then using (14)
and (15) we get

$$
\psi_{n+1}(a) \geq \int_{B \cap J_1} |\tau_{k+m}|d\mu > \zeta \int_{B \cap J} |\tau_k|d\mu = \zeta \psi_n(a)
$$

for each \( a \in B \cap J_1 \). Hence, we get the claim of this lemma.

Next, we suppose that \( B \cap J_1 \neq \emptyset \) and \( B \cap J_2 \neq \emptyset \). Then \( J_1 \) and \( J_2 \) are both immature but \((k + m)\)-useful. There exist \( a_0 \in \partial J \) and \( \tilde{y} \) such that \( \varphi_k(a_0) = (0, \tilde{y}) \in \{y\text{-axis}\} \). Then,

$$
\varphi_{m+k}(a_0) = f^{m+k}_{a_0}(0, y) = f^m_{a_0} \circ f^k_{a_0}(0, y) = f^m_{a_0}(\varphi_k(a_0)) = f^m_{a_0}(0, \tilde{y}).
$$

If \( m \in \{1, 2, \ldots, 9\} \), \( J_i \) is mature. This is a contradiction. Thus we have \( m \geq 10 \). From (15), for any \( m \geq 10 \), we get

$$
|\tau_{k+m}| > \zeta^{10}|\tau_k|.
$$

By the mean value theorem, there exists \( a^{(i)} \in B \cap J_i \) such that

$$
\int_{B \cap J_i} |\tau_{k+m}(a)|d\mu = |\tau_{k+m}(a^{(i)})|\mu(B \cap J_i).
$$

By Proposition 4, for \( k + m \geq N \), we have

$$
\frac{\int_{B \cap J_1} |\tau_{k+m}(a)|d\mu}{\int_{B \cap J_2} |\tau_{k+m}(a)|d\mu} \frac{\mu(B \cap J_2)}{\mu(B \cap J_1)} = \frac{|\tau_{k+m}(a^{(1)})|}{|\tau_{k+m}(a^{(2)})|} < 1 + \gamma.
$$

Then, using (13) we get

$$
\frac{\mu(B \cap J_2)}{\mu(B \cap J_1)} < 1 + \gamma.
$$

Since \( \mu(B \cap J) = \mu(B \cap J_1) + \mu(B \cap J_2) \), we have

$$
(2 + \gamma)\mu(B \cap J_1) > \mu(B \cap J).
$$

Hence, using (14), (16) and (17), we get

$$
\int_{B \cap J_1} \psi_{n+1}d\mu \geq \mu(B \cap J_1) \int_{B \cap J_1} |\tau_{k+m}|d\mu \\
\geq \frac{\mu(B \cap J)}{2 + \gamma} \cdot \frac{1}{2} \int_{B \cap J} |\tau_{k+m}|d\mu \\
\geq \frac{\mu(B \cap J)}{2(2 + \gamma)} \cdot \zeta^{10} \int_{B \cap J} |\tau_k|d\mu
$$
\[
\zeta^{10} = \frac{\zeta^{10}}{2(2 + \gamma)} \int_{B \cap J} \psi_n d\mu.
\]

Since \(\zeta > 1.15\), see Lemma 5, we have \(\zeta^{10} > 4\).

In Proposition 4, \(\gamma > 0\) can be arbitrarily small. Then, we have
\[
\zeta^{10} > 2(2 + \gamma)
\]
(18)

**Proposition 8** For almost every \(a \in I\), there exist \(k \geq N\) and open interval \(I' \subset I\) with \(a \in I'\) such that \(I'\) is \(k\)-mature.

**Proof.** We just show that \(\mu(B) = 0\). Let \(P_n\) be a partition of \(I\). Now suppose \(\mu(B') > 0\). Then, there exists some \(k\)-useful \(J \in P_n\) such that \(\mu(B \cap J) > 0\). Since \(\tau_k\) is the tangent vector of \(\varphi_k\), we have
\[
\psi_n(a) = \int_{B \cap J} |\tau_k| d\mu \leq \int_J |\tau_k| d\mu = \text{Length}(\varphi_k(J)),
\]
where \(\text{Length}(\varphi_k(J))\) is bounded by some constant \(K\) independent of \(n\) because of the trapping region. Then we have for all \(n \geq N\)
\[
\int_B \psi_n(a) d\mu < K \cdot \mu(B).
\]
(19)

By Lemma 7 and (18), there is \(\sigma > 1\) such that, for all \(n \geq N\),
\[
\int_B \psi_{n+1} d\mu \geq \sigma \int_B \psi_n d\mu.
\]
This means that \(\int_B \psi_n d\mu\) increases exponentially for \(n \geq N\), which contradicts (19). Then we have \(\mu(B) = 0\). \(\square\)

**Proposition 9** Let \(I' \subset I\) be a \(k\)-mature interval which is obtained for almost every \(a \in I\) in Proposition 8. Then, there exists \(\bar{a} \in \partial I'\) such that
\[
|\varphi_k(\bar{a})|_{x} \geq \frac{1}{2}.
\]

**Proof.** Since \(I'\) is \(k\)-mature, there exist \(\bar{a} \in \partial I'\) and \((0, \bar{y}) \in \mathcal{Y}_{a,b}\) such that
\[
\varphi_k(\bar{a}) = f^{\text{ma}}_{\bar{a}}(0, \bar{y})
\]
where $m \in \{1, 2, \ldots, 9\}$. By (4), we have the fact that
\[ f_a^m(0, \bar{y}) \not\in \{(x, y) \in \mathbb{R}^2 : |x| < \frac{1}{2}\}. \]
Then,
\[ \varphi_k(\bar{a}) \not\in \{(x, y) \in \mathbb{R}^2 : |x| < \frac{1}{2}\}. \]

\[ \square \]

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