H-projective-recurrent Kählerian manifolds and Bochner-recurrent Kählerian manifolds

By Izumi HASEGAWA

Introduction.

T. Adati and T. Miyazawa [1] investigated the conformal-recurrent Riemannian manifolds and M. Matsumoto [2] the projective-recurrent Riemannian manifolds. In their paper, they concerned with the more general Riemannian manifolds, that is, the Riemannian metric $g$ is not necessarily positive definite.


The purpose of the present paper is to make researches in the H-projective-recurrent Kählerian manifolds and the Bochner-recurrent Kählerian manifolds.

The present author wishes to express his hearty thanks to Professor Y. Katsurada for her many valuable advices and encouragement.

§ 1. Preliminaries.

Let $M$ be an $n(=2m)$ dimensional Kählerian manifold with Kählerian structure $(g, J)$ satisfying

$$ J'^{a}_{i}J^{a}_{j} = -\delta^{i}_{j}, \quad J_{i}^{j} = -J_{j}^{i}, \quad \nabla_{h}J_{i}^{j} = 0, $$

where $J_{i}^{j} = g_{i}{}^{a}J^{a}_{j}$.

It is well known that the tensor

$$ P_{hjk} = R_{hjk} - \frac{1}{n+2}(R_{ij}g_{hk} - R_{ik}g_{hj} + H_{ij}J_{hk} - H_{ik}J_{hj} - 2H_{ih}J_{jk}), $$

where $H_{ij} = R_{ij}J^{a}_{j}$, is called the holomorphically projective (for brevity, H-projective) curvature tensor of $M$, and the tensor

$$ B_{hjk} = R_{hjk} - \frac{1}{n+4}(R_{ij}g_{hk} - R_{ik}g_{hj} + H_{ij}J_{hk} - H_{ik}J_{hj} - 2H_{ih}J_{jk}) $$

$$ + R_{hjk}g_{ij} - R_{ik}g_{hj} + H_{hj}J_{ij} - H_{ij}J_{hj} - 2H_{ij}J_{jk} $$

$$ + \frac{R}{(n+2)(n+4)}(g_{ij}g_{hk} - g_{ik}g_{hj} + J_{ij}J_{hk} - J_{ik}J_{hj} - 2J_{ih}J_{jk}) $$
the Bochner curvature tensor of $M$.

We consider a tensor $U_{hifk}$ given by

$$U_{hifk} = \dot{R}_{hifk} - \frac{R}{n(n+2)}(g_{if}g_{hk} - g_{hf}g_{ik} + J_{i}J_{\backslash}J_{hk} - J_{hf}J_{ik} - 2J_{hi}J_{fk}).$$

Hence we call this tensor the $H$-concircular curvature tensor of $M$. The $H$-projective curvature tensor and the Bochner curvature tensor coincide with the $H$-concircular curvature tensor of $M$ if and only if $M$ is an Einstein space.

We call that a Kählerian manifold $M$ is $H$-projective-recurrent if $\nabla_{l}P_{hifk} = \kappa_{l}P_{hifk}$ where $\kappa_{l}$ is the vector of $H$-projective-recurrence, Bochner-recurrent $\nabla_{l}B_{hifk} = \kappa_{l}B_{hifk}$ where $\kappa_{l}$ is the vector of Bochner-recurrence and $H$-concircular-recurrent if $\nabla_{l}U_{hifk} = \kappa_{l}U_{hifk}$ where $\kappa_{l}$ is the vector of $H$-concircular-recurrence.

We call that a Kählerian manifold $M$ is $H$-projective-symmetric if the $H$-projective curvature tensor is parallel, that is, $\nabla_{l}P_{hifk} = 0$. Similarly, we define the Bochner-symmetric Kählerian manifold and $H$-concircular-symmetric Kählerian manifold.

We have well known the following identities:

$$g_{ab}J_{c}^{a}J_{f}^{b} = g_{if},$$
$$R_{ab}J_{c}^{a}J_{f}^{b} = R_{if},$$
$$R_{ia}J_{f}^{a} = -R_{fa}J_{i}^{a},$$
$$\nabla^{a}R_{aifk} = \nabla_{k}R_{if} - \nabla_{f}R_{ik},$$
$$\nabla_{k}R = 2\nabla_{a}R_{k}^{a},$$

(1. 5)

$$H_{ij} = -H_{ji},$$
$$H_{ab}J_{c}^{a} = R_{c},$$
$$H_{ia}J_{f}^{a} = J_{fa}J_{i}^{a} = -R_{if},$$
$$H_{ij} = -(1/2)R_{abc}J_{c}^{ab} = R_{abc}J_{c}^{ab},$$
$$\nabla_{a}H_{kj}J_{i}^{a} = \nabla_{k}R_{ij} - \nabla_{j}R_{ik},$$
$$\nabla_{a}R_{k}^{a} = 2\nabla_{a}H_{k}^{a}.$$  


**Theorem 1.** A necessary and sufficient condition for a Kählerian manifold $M$ to be $H$-projective-recurrent is that $M$ be $H$-concircular-recurrent.

**Proof.** We assume that a Kählerian manifold $M$ is $H$-concircular-recurrent, i.e.

$$\nabla_{l}U_{hifk} = \kappa_{l}U_{hifk}.$$  

From (1. 4), we can write (2. 1) as

$$(2. 1)^{*} \quad \nabla_{l}R_{hifk} = \kappa_{l}R_{hifk} + \frac{1}{n(n+2)}(\nabla_{l}R - \kappa_{l}R)\mathcal{A}_{hifk},$$
where \( \mathcal{A}_{hifk} = g_{if}g_{hk} - g_{hf}g_{ik} + J_{if}J_{hk} - J_{hf}J_{ik} - 2J_{hi}J_{fk} \).

Contracting (2.1)* with \( g^{hk} \), we get

\[
(2.2) \quad \mathcal{V}_l R_{ij} = \kappa_l R_{ij} + \frac{1}{n} (\mathcal{V}_l R - \kappa_l R) g_{ij} .
\]

Substituting (2.1)* and (2.2) in \( \mathcal{V}_l P_{hijk} \), we have

\[
(2.3) \quad \mathcal{V}_l P_{hijk} = \kappa_l P_{hijk} ,
\]

that is, \( M \) is H-projective-recurrent.

Conversely, we assume that \( M \) is H-projective-recurrent, than we have

\[
(2.3)^* \quad \mathcal{V}_l R_{hijk} = \kappa_l R_{hijk} + \frac{1}{n+2} \left\{ (\mathcal{V}_l R_{if} g_{hk} - \mathcal{V}_l R_{hf} g_{ik} + \mathcal{V}_l H_{ij} J_{hk} - \mathcal{V}_l H_{hj} J_{ik} - 2 \mathcal{V}_l H_{hi} J_{fk}) - \kappa_l (R_{if} g_{hk} - R_{hj} g_{ik} + H_{if} J_{hk} - H_{hj} J_{ik} - 2H_{hi} J_{fk}) \right\} .
\]

Transecting (2.3)* with \( g^{ij} \), we get

\[
(2.4) \quad \mathcal{V}_l R_{hk} = \kappa_l R_{hk} + \frac{1}{n} (\mathcal{V}_l R - \kappa_l R) g_{hk} ,
\]

Substituting this in (2.3)*, we obtain (2.1)*, i.e. (2.1). Q.E.D.

From Theorem 1, we have the following corollaries.

**Corollary 1.** If a H-projective-recurrent Kählerian manifold \( M \) satisfies \( \mathcal{V}_l R = \kappa_l R \), where \( \kappa_l \) is the vector of H-projective-recurrence, then \( M \) is recurrent.

**Corollary 2.** A necessary and sufficient condition for a Kählerian manifold \( M \) to be H-projective-symmetric is that \( M \) be H-concircular-symmetric.

**Corollary 3.** If a H-projective-symmetric Kählerian manifold \( M \) has the constant scalar curvature, then \( M \) is symmetric.

**Proposition 2.** If a Kählerian manifold \( M \) is H-projective-recurrent, then \( M \) satisfies the identity

\[
(2.4) \quad (n-2) \mathcal{V}_k R = 2n \kappa_k R^a_k - 2 \kappa_k R ,
\]

where \( \kappa_k \) is the vector of H-projective-recurrence.

**Proof.** Contracting (2.1)* with \( g^{ih} \), we get

\[
(2.5) \quad \mathcal{V}^a R_{atjk} = \kappa^a R_{atjk} + \frac{1}{n(n+2)} (\mathcal{V}^a R - \kappa^a R) \mathcal{A}_{atjk} ,
\]

where \( \kappa^a = g_{ab} \kappa_b . \)
Using (1.5) in the left side of (2.5), we obtain

\[(2.6)\]
\[\nabla_a H_{kj} J^a_i = \kappa^a R_{aijk} + \frac{1}{n(n+2)}(\nabla^a R - \kappa^a R) A_{aijk}.\]

Transvecting this with \(J^i_l J^m_l\), we get

\[(2.7)\]
\[\nabla_l R_{km} = \kappa^a R_{abck} J^b_m + \frac{1}{n(n+2)}(\nabla^a R - \kappa^a R)(g_{lm}g_{ak} + g_{am}g_{lk} + 2g_{al}g_{mk} + J_{lm}J_{ak} + J_{am}J_{lk}).\]

Moreover contracting this with \(g^{km}\), we obtain

\[\nabla_l R = 2\kappa^a R_{al} + \frac{2}{n}(\nabla_l R - \kappa_l R),\]

whence (2.4) follows. Q.E.D.

As an immediate consequence of this proposition and Corollary 3, we have the following

**Corollary 4.** In a H-projective-symmetric Kählerian manifold \(M\), the scalar curvature \(R\) is constant. Therefore \(M\) is symmetric.

Now, we assume that a Kählerian manifold \(M\) is H-projective-recurrent and \(M\) is not of constant holomorphic sectional curvature. We have

\[(2.8)\]
\[\nabla_l (P_{hifk} P^{hifk}) = 2\kappa_l (P_{hifk} P^{hifk}),\]

whence it follows that \(\kappa_l\) is gradient.

Using the Ricci identity and Theorem 1, we have the following

**Proposition 3.** A H-projective-recurrent Kählerian manifold \(M\) satisfies the condition \(\nabla_m \nabla_l R_{hifk} = \nabla_l \nabla_m R_{hifk}\).

Next, we have the following

**Theorem 4.** If a Kählerian manifold \(M\) is H-projective-recurrent, then \(M\) is recurrent.

**Proof.** We have the following two cases: (a) \(M\) is of constant holomorphic sectional curvature, (b) the vector of H-projective-recurrence \(\kappa_l\) is gradient. In the case (a), \(M\) is symmetric, whence it follows that \(M\) is recurrent.

Now, we shall consider with the case (b).

We consider a tensor \(U_{ij}\) given by

\[(2.9)\]
\[U_{ij} = R_{ij} - \frac{R}{n} g_{ij}.\]

In a H-projective-recurrent Kählerian manifold \(M\), from Theorem 1, we
have (2.1), whence we obtain

\[ V_t U_{ij} = \kappa_t U_{ij} . \]

Since \( \kappa_t \) is gradient, we have

\[ V_m V_t U_{ij} - V_t V_m U_{ij} = 0 . \]

Applying the Ricci identity to (2.11), we obtain

\[ 0 = R_{mitj} a U_{aj} + R_{mitj} a U_{ea} \]

\[ = U_{mitj} a U_{aj} + U_{mitj} a U_{ea} + \frac{R}{n(n+2)} (\mathscr{A}_{mitj} a U_{aj} + \mathscr{A}_{mitj} a U_{ea}) . \]

Differentiating this covariantly, we get

\[ 0 = 2\kappa_p (U_{mitj} a U_{aj} + U_{mitj} a U_{ea}) \]

\[ + \frac{1}{n(n+2)} (V_p R + \kappa_p R)(\mathscr{A}_{mitj} a U_{aj} + \mathscr{A}_{mitj} a U_{ea}) . \]

It follows from (2.12) and (2.13) that

\[ (V_p R - \kappa_p R)(\mathscr{A}_{mitj} a U_{aj} + \mathscr{A}_{mitj} a U_{ea}) = 0 . \]

Contracting this with \( g^{ij} \), we obtain

\[ (V_p R - \kappa_p R) U_{mj} = 0 . \]

Thus we find either \( V_p R = \kappa_p R \) or \( U_{mj} = 0 \).

In the case \( V_p R = \kappa_p R \), from Corollary 1, \( M \) is recurrent.

In the case \( U_{ij} = 0 \), \( M \) is symmetric. (see §3. Theorem 6 or [3]) Q.E.D.

§ 3. Bochner-recurrent Kählerian manifolds.

It is clear that a Bochner-recurrent Kählerian manifold satisfying the condition \( V_k R_{ij} = \kappa_k R_{ij} \), where \( \kappa_k \) is the vector of Bochner-recurrence, is recurrent.

In this section, first, we shall prove the following

THEOREM 5. In order that a Bochner-recurrent Kählerian manifold \( M \) is \( H \)-projective-recurrent, it is necessary and sufficient to be \( V_k R_{ij} = \kappa_k R_{ij} + \frac{1}{n} (V_k R - \kappa_k R) g_{ij} \), where \( \kappa_k \) is the vector of Bochner-recurrence.

PROOF. We assume that a Kählerian manifold \( M \) is \( H \)-projective-recurrent, then from the proof of Theorem 1 we have (2.1)* and (2.2). Substituting (2.1)* and (2.2) in \( V_t B_{mjk} \), we have

\[ V_t B_{mjk} = \kappa_t B_{mjk} . \]
Conversely, we assume that a Bochner-recurrent Kählerian manifold $M$ satisfies the condition (2.2) where $\kappa_i$ is the vector of Bochner-recurrence, then we have

$$\nabla_l R_{hifk} = \kappa_l R_{hiff} + \frac{1}{n+4} \left\{ \nabla_l (\mathcal{B}_{hifk} + \mathcal{B}_{ihfk} - 2H_{hi}J_{jk} - 2H_{fk}J_{hi}) - \kappa_l (-\mathcal{B}_{hifk} + \mathcal{B}_{ihfk} - 2H_{hi}J_{fk} - 2H_{fk}J_{hi}) \right\} - \frac{1}{(n+2)(n+4)} (\nabla_l R - \kappa_l R) \mathcal{A}_{hifk},$$

where $\mathcal{B}_{hifk} = R_{i[f}g_{hk]} - R_{h[f}g_{i}k]} + H_{i[f}J_{hk]} - H_{h[f}J_{i}k]$.

Substituting (2.2) in (3.1)*, we have (2.1)*, that is, (2.1). From Theorem 1, $M$ is H-projective-recurrent. Q.E.D.

**Theorem 6.** If a Bochner-recurrent Kählerian manifold $M$ is Ricci-symmetric, then either the Bochner curvature tensor vanishes or the vector of Bochner-recurrence is zero. Consequently $M$ is symmetric.

**Proof.** If a Bochner-recurrent Kählerian manifold $M$ is Ricci-symmetric, we have

$$\nabla_l R_{hifk} = \kappa_l B_{hifk}.$$

From the Bianchi's identity and (3.2), we get

$$\kappa_l B_{hifk} + \kappa_h B_{ilfk} + \kappa_i B_{lhfk} = 0.$$

Transvecting (3.3) with $\kappa^i$, we have

$$\kappa_l \kappa^i B_{hifk} + \kappa_h \kappa^i B_{ilfk} + \kappa_i \kappa^i B_{lhfk} = 0.$$

Since $\nabla^a R_{hij} = \nabla_a R_{ij} - \nabla_j R_{ai} = 0$, we have $\kappa^i B_{iij} = 0$ and $\kappa^i B_{iij} = 0$.

Now, we obtain $(\kappa_l \kappa^i) B_{hifk} = 0$. Consequently, $\kappa_l$ is zero or the Bochner curvature tensor vanishes. Therefore $M$ is symmetric. Q.E.D.

**Theorem 7.** If a Kählerian manifold $M$ is Bochner-recurrent and Ricci-recurrent, then $M$ is recurrent.

**Proof.** We assume that the Bochner curvature tensor does not vanish in a Bochner-recurrent Kählerian manifold $M$. Then the vector of Bochner-recurrence $\kappa_i$ is gradient.

We put $\kappa^* i$ the vector of Ricci-recurrence and

$$\mathcal{C}_{hifk} = R_{hifk} - B_{hifk},$$

whence it follows that

$$\nabla_l R_{hifk} = \kappa_l B_{hifk} + \kappa^* i \mathcal{C}_{hifk}.$$

1) This theorem was proved by T. Yamada.
Since either $B_{hifj}=0$ or $\kappa_i$ is gradient, we have

\[(3.7) \quad \nabla_m \nabla_l B_{hifj} - \nabla_l \nabla_m B_{hifj} = 0.\]

Using the Ricci identity to (3.7), we obtain

\[(3.8) \quad 0 = R_{mlh}^a B_{aifj} + R_{mli}^a B_{hajk} + R_{mlj}^a B_{hia} + R_{mlk}^a B_{hjia} + \mathcal{C}_{mlh}^a B_{aifj} + \mathcal{C}_{mli}^a B_{hajk} + \mathcal{C}_{mlj}^a B_{hiak} + \mathcal{C}_{mlk}^a B_{hjfa}.\]

The covariant differentiation of (3.8) gives

\[(3.9) \quad 0 = 2\kappa_p (B_{mlh}^a B_{aifj} + B_{mli}^a B_{hajk} + B_{mlj}^a B_{hia} + B_{mlk}^a B_{hjia}) + (\kappa_p + \kappa^*)_p (\mathcal{C}_{mlh}^a B_{aifj} + \mathcal{C}_{mli}^a B_{hajk} + \mathcal{C}_{mlj}^a B_{hiak} + \mathcal{C}_{mlk}^a B_{hjfa}).\]

It follows from (3.8) and (3.9), that

\[(3.10) \quad (\kappa_p - \kappa^*)_p (\mathcal{C}_{mlh}^a B_{aifj} + \mathcal{C}_{mli}^a B_{hajk} + \mathcal{C}_{mlj}^a B_{hiak} + \mathcal{C}_{mlk}^a B_{hjfa}) = 0.\]

In the case $\kappa_p = \kappa^*_p$, clearly, $M$ is recurrent.

Next, we assume that

\[(3.11) \quad \mathcal{C}_{mlh}^a B_{aifj} + \mathcal{C}_{mli}^a B_{hajk} + \mathcal{C}_{mlj}^a B_{hiak} + \mathcal{C}_{mlk}^a B_{hjfa} = 0.\]

Contracting this with $g^{lh}$, we have

\[(3.12) \quad R_m^a B_{aifj} = 0.\]

Transvecting this with $\kappa^*_m$, we obtain

\[(3.13) \quad 0 = \kappa^*_b R_b^a B_{aifj} = \frac{R}{2} \kappa^* a B_{aifj}.\]

Thus, we find either

\[(3.14) \quad \kappa^* a B_{aifj} = 0\]

or $R=0$.

In the case (3.14), transvecting (3.11) with $\kappa^* l \kappa^*$, we obtain $\kappa^* l \kappa^* \mathcal{C}_{mlh}^a B_{aifj} = 0$, whence it follows that

\[(3.15) \quad R(\kappa^*_a \kappa^* a) B_{aifj} = 0.\]

In the case $B_{hifj} = 0$, we have

\[
\begin{align*}
\nabla_l R_{hifj} &= \nabla_l \mathcal{C}_{hifj} \\
&= \kappa^*_l \mathcal{C}_{hifj} \\
&= \kappa^*_l R_{hifj},
\end{align*}
\]
that is, \( M \) is recurrent.

Next, we shall consider the base \( \kappa^*_k = 0 \).

In this case, from Theorem 6, \( M \) is symmetric.

Finally, we shall consider the case \( R = 0 \).

Transvecting (3.11) with \( R^{th} \) and using (3.12), we have

\[
0 = (R^{ab}R_{ab})B_{mifjk} + R^{dr}R_{cb}J^b_{m}J^c_{a}B_{aifjk}
\]

(3.16)

\[
= (R^{ab}R_{ab})B_{mifjk}.
\]

Consequently \( M \) is recurrent.

Q.E.D.

References


(Received March 11, 1974)