Abstract. New functional inequalities are obtained for the capacities of Grötzsch and Teichmüller rings, and for complete elliptic integrals, thus solving two conjectures. These results are applied to refine earlier estimates in quasiconformal Schwarz lemma and Mori’s theorem.

Key words: conformal capacity, distortion, modulus, quasiconformal, ring.

1. Introduction

For \( n \geq 2, \ s > 1 \) and \( t > 0 \), let \( R_{G,n}(s) \) denote the Grötzsch ring in \( \mathbb{R}^n \), whose complementary components are the closed unit ball \( \overline{B}^n \) and the ray \([s, \infty)\) along the \( x_1 \)-axis, and let \( R_{T,n}(t) \) denote the Teichmüller ring in \( \mathbb{R}^n \), whose complementary components are the segment \([-1, 0]\) and the ray \([t, \infty)\) along the \( x_1 \)-axis. The conformal capacities of \( R_{G,n}(s) \) and \( R_{T,n}(t) \) are denoted by

\[
\gamma_n(s) = \text{cap} R_{G,n}(s), \quad \tau_n(t) = \text{cap} R_{T,n}(t), \quad (1.1)
\]

respectively. The modulus \( M_n(r) \) of the Grötzsch ring \( R_{G,n}(1/r) \), \( 0 < r < 1 \), is defined by

\[
M_n(r) = \left[ \frac{\omega_{n-1}}{\gamma_n(1/r)} \right]^{1/(n-1)}, \quad (1.2)
\]

where \( \omega_{n-1} \) is the \((n - 1)\)-dimensional measure of the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \) [G], [Vä], [Vu]. The capacities in (1.1) are related [G, §18] by

\[
\gamma_n(s) = 2^{n-1} \tau_n(s^2 - 1), \quad s > 1. \quad (1.3)
\]

For \( K > 0 \), define the increasing homeomorphism \( \varphi_{K,n}(r) \) from \([0, 1]\) onto \([0, 1]\) by

\[
\varphi_{K,n}(r) = \frac{1}{\gamma_n^{-1}(K \gamma_n(1/r))} = \frac{1}{M_n^{-1}(\alpha M_n(r))} \quad (1.4)
\]

for \( r \in (0, 1), \ \varphi_{K,n}(0) = \varphi_{K,n}(1) - 1 = 0, \) where \( \alpha = K^{1/(1-n)} \). These functions arise in the study of quasiconformal mappings in \( \mathbb{R}^n \) [AVV].

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As usual, for $n = 2$, we let
\[ \mu(r) = M_2(r), \quad \tau(t) = \tau_2(t) \quad \text{and} \quad \varphi_K(r) = \varphi_{K,2}(r). \] (1.5)

It is well-known that $\mu(r)$ has the explicit expression [LV, p. 60]
\[ \mu(r) = \frac{\pi}{2} \frac{\mathcal{K}'(r)}{\mathcal{K}(r)}, \] (1.6)
where
\[ \mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 x)^{-1/2} \, dx, \]
\[ \mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r'), \]
\[ r' = \sqrt{1 - r^2}, \] (1.7)
are complete elliptic integrals of the first kind \[ \text{BF}, \text{Bo}, \text{BB}. \]

Let $D$ and $D'$ be domains in $\mathbb{R}^n$, and for $K \geq 1$, $f : D \to D'$ be a $K$-quasiconformal mapping. The inner dilatation $K_I(f)$ and the outer dilatation $K_O(f)$ of $f$ are defined by [Vä], [Vu]
\[ K_I(f) = \sup \frac{M(f(\Gamma))}{M(\Gamma)}, \quad K_O(f) = \sup \frac{M(\Gamma)}{M(f(\Gamma))}, \] (1.8)
respectively, where $M(\Gamma)$ is the modulus of the curve family $\Gamma$. The suprema are taken over all curve families $\Gamma$ in $D$ such that $M(\Gamma)$ and $M(f(\Gamma))$ are not simultaneously 0 or $\infty$. Then $1 \leq K_I(f), K_O(f) \leq K$.

Recently, many inequalities were obtained for the above-mentioned special functions and several conjectures were raised [AVV3]–[AVV11], [QVV], [VV]. Among these conjectures, the following appear in [AVV7] and [AVV9]:

\begin{itemize}
  \item [(C_1)] For each $K \in (1, \infty)$, the function $f(r) = r^{-1/K} \varphi_K(r)$ is convex on $(0, 1)$.
  \item [(C_2)] $\frac{3}{2} \tau(t) < \tau(2t) + \tau(\sqrt{2t}) < 3\tau(t)$, for all $t \in (0, \infty)$.
\end{itemize}

On the other hand, some of the function-theoretic applications of the above-mentioned functions rely on the monotonicity and explicit estimates of these functions. An example of such an explicit estimate is [W], [LV], [He]
\[ r^{1/K} < \varphi_K(r) < 4^{1-1/K} r^{1/K}, \] (1.9)
for all $K > 1$ and $r \in (0, 1)$, from which the well-known explicit quasicon-
formal Schwarz lemma follows, that is,

\[ |f(z)| \leq 4^{1-1/K}|z|^{1/K}, \tag{1.10} \]

for each \(K\)-quasiconformal mapping \(f\) of the unit disk \(B\) into itself with \(f(0) = 0\) and for \(z \in B\). In [QVV, Theorem 1.5], the upper bound in (1.9) was improved to

\[ \varphi_K(r) < (1 + r')^{2(1-1/K)}r^{1/K}, \tag{1.11} \]

for \(r \in (0,1)\) and \(K > 1\), where \(r' = \sqrt{1-r^2}\), and hence, the inequality (1.10) was sharpened to

\[ |f(z)| \leq (1 + \sqrt{1-|z|^2})^{2(1-1/K)}|z|^{1/K}, \tag{1.12} \]

for all \(z \in B\).

In this paper we obtain some new functional inequalities for \(\mu(r)\) and \(\tau_n(t)\), prove conjecture (C_2), disprove conjecture (C_1), and sharpen the upper bound for \(\varphi_K(r)\), by studying some properties of complete elliptic integrals \(\mathcal{K}(r)\), \(\mathcal{K}'(r)\) in (1.7) and those of the second kind

\[ \mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 x)^{1/2} dx, \quad \mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r'), \tag{1.13} \]

for \(r \in (0,1)\), where \(r' = \sqrt{1-r^2}\). These inequalities lead to sharper estimates in the quasiconformal Schwarz lemma and Mori’s theorem.

Throughout this paper, \(r'\) denotes \(\sqrt{1-r^2}\), for \(r \in [0,1]\).

We now state some of the main results of this paper.

**Theorem 1.14** Conjecture (C_1) is false.

**Theorem 1.15** The function \(F(t) = [\tau(2t) + \tau(\sqrt{2t})]/\tau(t)\) is strictly increasing from \((0, \infty)\) onto \((\frac{3}{2}, 3)\). In particular, for all \(t \in (0, \infty)\),

\[ \frac{3}{2}\tau(t) < \tau(2t) + \tau(\sqrt{2t}) < 3\tau(t), \tag{1.16} \]

so Conjecture (C_2) is true.

**Theorem 1.17** For \(n \geq 2, p \in (0, \infty)\), let \(C_n(p) = p^{-1/4}\), if \(n = 2\), and \(= p^{-1/2}\), if \(n \geq 3\). Then
(1) For each $t \in (0, \infty)$,
\[
C_n(p) < \frac{\tau_n(pt)}{\tau_n(t)} < 1, \quad \text{if} \quad 1 < p < \infty,
\]
\[
1 < \frac{\tau_n(pt)}{\tau_n(t)} < C_n(p), \quad \text{if} \quad 0 < p < 1.
\] (1.18)

(2) For $t \in (0, \infty)$ and $p \geq 1$,
\[
\left(1 + \frac{1}{p}\right) C_n(p) \tau_n(t) \leq \tau_n(pt) + \tau_n((pt)^{1/p}) \leq (1 + p^{n-1}) \tau_n(t).
\] (1.20)

(3) For $p > 1$ and $t \geq p^{1/(p-1)},$
\[
\tau_n(pt) + \tau_n((pt)^{1/p}) \geq (1 + C_n(p)) \tau_n(t).
\] (1.21)

Remark 1.22. For $n = 2$, Theorem 1.17 (1) improves [AVV8, Theorem 1.8 (3) and (4)].

Theorem 1.23 For $r \in (0, 1)$, define the function $g$ on $[1, \infty)$ by
\[
g(K) = \varphi_K(r)[(1 + r')^{1+r'}/r]^{1/K}.
\]
Then $g$ is strictly decreasing from $[1, \infty)$ onto $(1, (1+r')^{1+r'})$. In particular, for $r \in (0, 1)$ and $K > 1$,
\[
\varphi_K(r) < (1 + r')^{(1+r')(1-1/K)} r^{1/K}.
\] (1.24)

Theorem 1.25 For $K \geq 1$, let $f$ be a $K$-quasiconformal automorphism of the closed unit disk $\overline{B}$ with $f(0) = 0$.
(a) For each $z \in \overline{B}$,
\[
|f(z)| \leq (1 + u)^{(1+u)(1-1/K)}|z|^{1/K},
\] (1.26)
where $u = \sqrt{1-|z|^2}$.
(b) For $z_1, z_2 \in \overline{B}$, let $v = (|z_1| + |z_2|)/2$, $w = |z_1 - z_2|/(4v)$ and $t = 2w$. Then
\[
|f(z_1) - f(z_2)| \leq C(z_1, z_2)^{1-1/K} |z_1 - z_2|^{1/K} \leq 64^{1-1/K} |z_1 - z_2|^{1/K},
\] (1.27)
where \( C(z_1, z_2) = 4(1 + v')^{1+w'}(1 + w')^{1+w'}\sqrt{(1 + t')/2} \).

(c) For \(|z_1| = |z_2| = 1\),

\[
|f(z_1) - f(z_2)| \leq \left[8(1 + r')^{2(1+r')}(1 + s')\right]^{(1-1/K)/2}|z_1 - z_2|^{1/K}
\]

\[
\leq 16^{1-1/K}|z_1 - z_2|^{1/K},
\]

where \( r = |z_1 - z_2|/4 \) and \( s = 2r \).

## 2. Preliminary Results

To prove the main theorems stated in Section 1, we need some functional inequalities for complete elliptic integrals. In this section, we study some monotone properties of some functions defined in terms of complete elliptic integrals, from which several functional inequalities follow. We apply these results to derive sharp inequalities for the special functions \( \mu(r) \), \( \tau_n(t) \) and \( \varphi_{K}(r) \).

The following well known relations will be useful in our proofs [BF], [He], [AVV10, Lemma 2.1].

\[
\frac{d\mathcal{K}}{dr} = \frac{\mathcal{E} - r'r^2\mathcal{K}}{rr^2}, \quad \frac{d\mathcal{E}}{dr} = \frac{\mathcal{E} - \mathcal{K}}{r},
\]

(2.1)

\[
\mathcal{K}\mathcal{E}' + \mathcal{K}'\mathcal{E} - \mathcal{K}\mathcal{K}' = \frac{\pi}{2},
\]

(2.2)

\[
\frac{d\mu}{dr} = -\frac{\pi^2}{4}, \frac{1}{rr^2\mathcal{K}^2(r)},
\]

(2.3)

\[
\frac{\partial s}{\partial r} = \frac{1}{K}, \frac{ss'r^2\mathcal{K}^2(s)}{rr^2\mathcal{K}^2(r)},
\]

(2.4)

\[
\frac{\partial s}{\partial K} = \frac{2}{\pi K}, \frac{ss'r^2\mathcal{K}(s)\mathcal{K}'(s)}{rr^2\mathcal{K}^2(r)},
\]

(2.5)

where \( s = \varphi_{K}(r) \), \( 0 < r < 1 \), \( 0 < K < \infty \).

**Lemma 2.6** For \( p > 1 \), \( t \in (0, \infty) \), let \( x = \sqrt{tp/(tp + 1)} \) and \( y = \sqrt{t/(t+1)} \). Then, as a function of \( t \), \( f(t) = p\mathcal{K}(y)\mathcal{K}'(y) - \mathcal{K}(x)\mathcal{K}'(x) \) is strictly increasing on \((0,1]\) and decreasing on \([1, \infty)\), with

\[
f((0, 1]) = f([1, \infty)) = ((p - 1)\pi \log 2, (p - 1)\mathcal{K}^2(1/\sqrt{2})].
\]
Proof. Clearly, \( f(1/t) = f(t) \). Hence, it is enough to prove the result only for \( t \in (0, 1] \), so that \( 0 < x \leq y \leq 1/\sqrt{2} \), with equalities if and only if \( t = 1 \).

Next, since \([Bo]\)

\[
\mathcal{K}(r) = \log(4/r') + O(r'^2 \log r')
\]  

(2.7)
as \( r \) tends to 1, it follows that \( f(0^+) = (p - 1)\pi \log 2 \). Clearly, \( f(1) = (p - 1)\mathcal{K}^2(1/\sqrt{2}) \).

Now, differentiation gives

\[
\frac{dx}{dt} = \frac{px^2 r'}{2t}, \quad \frac{dy}{dt} = \frac{y y'^2}{2t}
\]

so that

\[
 tf'(t) = p[h(y) - h(x)]
\]

by \([2.2]\), where \( h(r) = \mathcal{K}'(r)[\mathcal{E}(r) - r'^2 \mathcal{K}(r)] \). Hence, \( f'(t) > 0 \) for \( t \in (0, 1) \) by \([AVV10, p. 539]\), yielding the result. \( \square \)

**Corollary 2.8**  
(a) For \( p > 1 \), \( t \in (0, \infty) \), let \( x = \sqrt{t^p/(t^p + 1)} \) and \( y = \sqrt{t/(t + 1)} \). Then \( f_1(t) = \mu(y)/\mu(x) \) is strictly increasing from \((0, \infty)\) onto \((1/p, p)\). In particular,

\[
\frac{1}{p} \mu(x) < \mu(y) < p \mu(x).
\]  

(2.9)

(b) For each \( p > 1 \), the function \( f_2(t) = \tau(t)/\tau(t^p) \) is strictly increasing from \((0, \infty)\) onto \((1/p, p)\). In particular, for all \( t \in (0, \infty) \),

\[
\frac{1}{p} \tau(t) < \tau(t^p) < p \tau(t).
\]  

(2.10)

Proof.  
(a) By l'Hospital's Rule, \( f_1(0^+) = 1/p \) and \( f_1(\infty) = p \).

Next, by differentiation,

\[
\frac{4}{\pi} t \mathcal{K}'^2(x) \mathcal{K}^2(y) f'_1(t) = f(t),
\]

where \( f \) is as in \([\text{Lemma 2.6}]\), and the result follows.

(b) Put \( x = \sqrt{t^p/(t^p + 1)} \) and \( y = \sqrt{t/(t + 1)} \). Then \( f_2(t) = f_1(t) \) by \([1.2]\) and \([1.3]\). Now the result follows from (a). \( \square \)
**Corollary 2.11** For $r \in (0, 1)$, $K \in [1, \infty)$, $p \in [1, \infty)$ and $c = 2/p$,\
\[
\frac{\varphi_K^c(r)}{\varphi_K^c(r) + \varphi_{1/K}^c(r')} \leq \varphi_K^2(\sqrt{\frac{r^c}{r^c + r'^c}}).
\]

(2.12)\

The inequality is reversed if $0 < K \leq 1$, with equality iff $p = 1$ or $K = 1$.\

**Proof.** By Corollary 2.8 (a), as a function of $K$, $\mu(s) = s^c/(s^c + s^{c'})$ is strictly increasing on $[1, \infty)$, and hence,
\[
\mu\left(\sqrt{\frac{s^c}{s^c + s'^c}}\right)/\mu(s) \geq \mu\left(\sqrt{\frac{r^c}{r^c + r'^c}}\right)/\mu(r),
\]

from which (2.12) follows. The remaining conclusions are clear. \qed

**Lemma 2.13** For $p > 1$ and $t \in (0, \infty)$, let $x = \sqrt{t/(t+1)}$ and $y = \sqrt{pt/(pt+1)}$. Then, as a function of $t$, $f(t) \equiv \mathcal{K}(y)\mathcal{K}'(y) - \mathcal{K}(x)\mathcal{K}'(x)$ is strictly increasing from $(0, \infty)$ onto $(-\frac{\pi}{4} \log p, \frac{\pi}{4} \log p)$.\

**Proof.** Clearly, $p(x/x')^2 = (y/y')^2$, $\frac{dx}{dt} = \frac{xx'^2}{2t}$ and $\frac{dy}{dt} = \frac{yy'^2}{2t}$. Hence, differentiation and (2.2) give
\[
 tf'(t) = h(y) - h(x) > 0
\]
by [AVV10, p. 539], where $h(r) = \mathcal{K}'(r)[\mathcal{E}(r) - r'^2\mathcal{K}(r)]$. This yields the monotoneity of $f$.\n
By (2.7), $f(0^+) = -f(\infty) = -\frac{\pi}{4} \log p$. \qed

**Corollary 2.15** (a) For $p > 1$, $t \in (0, \infty)$, let $x = \sqrt{t/(t+1)}$ and $y = \sqrt{pt/(pt+1)}$. Then $g_1(t) = \mu(y)/\mu(x)$ is strictly decreasing on $(0, 1/\sqrt{p})$ and increasing on $[1/\sqrt{p}, \infty)$ with $g_1(0^+) = g_1(\infty) = 1$. In particular, for $p > 1$ and $t \in (0, \infty)$, $u^2 = 1/(1 + \sqrt{p})$,
\[
p^{-1/4} < \left(\frac{\mathcal{K}(u)}{\mathcal{K}'(u)}\right)^2 \leq \frac{\mu(y)}{\mu(x)} < 1,
\]

(2.16)\

with equality iff $t = 1/\sqrt{p}$.\

(b) For each $p > 1$, the function $g_2(t) = \tau(pt)/\tau(t)$ is strictly decreasing on $(0, 1/\sqrt{p})$ and increasing on $[1/\sqrt{p}, \infty)$ with $g_2(0^+) = g_2(\infty) = 1$.\n
In particular, for $p > 1$ and $t \in (0, \infty)$, $u^2 = 1/(1 + \sqrt{p})$,

$$p^{-1/4} < \left( \frac{K(u)}{K'(u)} \right)^2 \leq \frac{\tau(pt)}{\tau(t)} < 1,$$

(2.17)

with equality iff $t = 1/\sqrt{p}$.

Proof. (a) The first inequality in (2.16) follows from [AVV10, Theorem 2.2 (3)]. Next, since $g_1(t) = g_1(1/(pt))$ by the relation [LV]

$$\mu(r)\mu(r') = \pi^2/4,$$

(2.18)

we need only prove the result (a) for $t \in (0, 1/\sqrt{p})$.

Differentiation and (2.2) give

$$\frac{4}{\pi} t (K'(x)K(y))^2 g_1'(t) = f(t) < f(1/\sqrt{p}) = 0,$$

where $f$ is as in Lemma 2.13, yielding the monotonicity of $g_1$.

By l'Hospital's Rule, $g_1(0^+) = g_1(\infty) = 1$. The second and third inequalities in (2.16) follow.

The result (b) follows from (a) and the relations (1.2) and (1.3). \(\square\)

In [QVV, Lemma 2.1], it was shown that

$$\frac{2}{\pi} r'^2 K(r)K'(r) + \log r < 2 \log(1 + r'),$$

(2.19)

for $r \in (0, 1)$, from which the inequality (1.12) was established. For further improvement of the inequality (1.12), we need to improve the estimate (2.19).

**Lemma 2.20** Let $m(r) = \frac{2}{\pi} r'^2 K(r)K'(r)$. Then there exists a unique $r_0 \in (\sin 55^\circ, \sin 56^\circ)$ such that the function $f(r) = m(r) + \log r - (1 + r') \log(1 + r')$ is strictly decreasing on $(0, r_0)$ and increasing on $[r_0, 1)$. In particular, for $r \in (0, 1)$,

$$-0.196 < f(r_0) \leq f(r) < 0,$$

(2.21)

with equality iff $r = r_0$.

Proof. By differentiation,

$$\frac{r'}{r} f'(r) = f_1(r) = 1 + \log(1 + r') - \frac{4}{\pi} K'(r) \cdot r' \frac{K(r) - \mathcal{E}(r)}{r^2}. $$

(2.22)
Let \( g_1(r) = \mathcal{K}(r) - \mathcal{E}(r) \) and \( g_2(r) = r^2/r' \). Then
\[
\frac{g_1'(r)}{g_2'(r)} = \frac{r'}{1 + r'^2} \mathcal{E}(r),
\]
which is strictly decreasing on \((0, 1)\). Hence, the function \( f_2(r) = r'[\mathcal{K}(r) - \mathcal{E}(r)]/r^2 \) is strictly decreasing from \((0, 1)\) onto \((0, \pi/4)\) by Monotone l'Hospital's Rule [AVV5, Lemma 2.2]. Hence, for \( r \in [a, b] \subset (0, 1) \),
\[
f_1(r) \leq f_3(a, b) = 1 + \log(1 + a') - \frac{4}{\pi b'} \mathcal{K}'(b) \frac{\mathcal{K}(b) - \mathcal{E}(b)}{b^2}.
\]
Since \( f_3(0, \sin 42^\circ) = -0.057127 \ldots \),
\[
f_1(r) < 0 \quad \text{for} \quad r \in (0, \sin 42^\circ]. \tag{2.23}
\]
Next, let \( f_4(r) = r^2 f_1(r) \). Then, \( f_4(r) = r^2 + 2r' + r^2 \log(1 + r') - \frac{4}{\pi} r' \mathcal{K}(r) \mathcal{E}'(r) \), by (2.2). By differentiation,
\[
\frac{r'}{r} f_4'(r) = 3r' - 3 + 2r' \log(1 + r') + \frac{4}{\pi} \mathcal{K}(r) \mathcal{E}'(r)
\]
\[
- \frac{4}{\pi} \frac{1}{r^2} [\mathcal{E}(r) \mathcal{E}'(r) - \mathcal{K}(r) \mathcal{E}'(r) + r^2 \mathcal{K}(r) \mathcal{K}'(r)].
\]
Clearly, \( 3r' - 3 + 2r' \log(1 + r') \) is decreasing on \((0, 1)\). Hence, by [QV, Theorem 2.1 (7)], it follows that
\[
\frac{r'}{r} f_4'(r) \geq f_5(a, b) = 3(b' - 1) + 2b' \log(1 + b')
\]
\[
+ \frac{4}{\pi a^2} \{ (1 + a^2) \mathcal{K}(a) \mathcal{E}'(a) - \mathcal{E}(a) \mathcal{E}'(a) - a^2 \mathcal{K}(a) \mathcal{K}'(a) \},
\]
for \( r \in [a, b] \subset (0, 1) \). By computation, we have
\[
f_5(\sin 73^\circ, 1) = 0.102028 \ldots, \quad f_5(\sin 63^\circ, \sin 73^\circ) = 0.052752,
\]
\[
f_5(\sin 55^\circ, \sin 63^\circ) = 0.0364292, \quad f_5(\sin 48^\circ, \sin 55^\circ) = 0.018148,
\]
and
\[
f_5(\sin 42^\circ, \sin 48^\circ) = 0.001202.
\]
Hence, \( f_4'(r) > 0 \) for \( r \in [\sin 42^\circ, 1) \), and \( f_4 \) is strictly increasing on \([\sin 42^\circ, 1)\). Since \( f_4(\sin 55^\circ) = -0.005950 < 0 \) and \( f_4(\sin 56^\circ) = 0.002438 > 0 \),
0, $f_4$ has a zero $r_0 \in (\sin 55^\circ, \sin 56^\circ)$ such that

$$f_4(r) \begin{cases} < 0, & \text{if } 0 < r < r_0, \\ > 0, & \text{if } r_0 < r < 1. \end{cases}$$

(2.24)

It follows from (2.23) and (2.24) that, on $(0, 1)$, $f_1$ has a unique zero $r_0$ such that $f_1(r) < 0$ for $r \in (0, r_0)$ and $f_1(r) > 0$ for $r \in (r_0, 1)$. This yields the piecewise monotonicity of $f$, so that the second and third inequalities in (2.21) follow.

Finally, since $m(r) + \log r$ is decreasing on $(0, 1)$ [AVV10, Lemma 4.2 (1)],

$$f(r_0) > m(\sin 56^\circ) + \log \sin 56^\circ - (1 + \cos 55^\circ) \log(1 + \cos 55^\circ)$$

$$= -0.195968 > -0.196.$$ This yields the first inequality in (2.21). \[\square\]

**Remark 2.25.** By [Lemma 2.20], it is natural to ask whether the function $f(r) = m(r) + \log r - (1 + r') \log(1 + r')$ is convex on $(0, 1)$. The answer to this question, however, is negative. As a matter of fact, differentiation gives

$$r'^3 f''(r) = F(r) = 1 + \log(1 + r') - \frac{r^2 r'}{1 + r'}$$

$$+ \frac{4}{\pi} r' \mathcal{K}' \left\{ \left( \frac{\mathcal{E}'}{\mathcal{K}'} + 1 - 2r^2 \right) \frac{\mathcal{K} - \mathcal{E}}{r^2} - \mathcal{E} \right\}.$$ Clearly, $F(0) = -\infty$ and $F(1) = 1$. Hence, there exist $r_1$ and $r_2$ with $0 < r_1 < r_2 < 1$ such that $f$ is concave on $(0, r_1)$ and convex on $(r_2, 1)$.

### 3. Proofs of Main Results

In this section, making use of the lemmas established in Section 2, we prove the theorems stated in Section 1.

**3.1. Proof of Theorem 1.14.** Let $s = \varphi_K(r)$. Then, by (2.4),

$$K f'(r) = \frac{1}{r} [g(r, K) - 1] f(r),$$

(3.2)

$$\frac{K r^2 f''(r)}{f(r)} = \frac{[g(r, K) - 1]^2}{K} + r \frac{\partial g}{\partial r} - g(r, K) + 1$$

$$\equiv f_1(r, K),$$

(3.3)
where
\[ g(r, K) = \left\{ s' \mathcal{K}(s)/(r' \mathcal{K}(r)) \right\}^2, \]
\[ r \frac{\partial g}{\partial r} = \frac{2g(r, K)}{r'^2 \mathcal{K}(r) \mathcal{K}'(r)} \left[ (r')^2 \mathcal{K}'(r) - (s')^2 \mathcal{K}'(s) \right]. \]

By l'Hospital’s Rule, we have
\[ \lim_{K \to 1} \frac{\mathcal{K}(r) \mathcal{E}'(r) - \mathcal{K}(s) \mathcal{E}'(s)}{K - 1} = -\frac{2}{\pi} \mathcal{K}(r) \mathcal{K}'(r) \left\{ \mathcal{E}(r) \mathcal{E}'(r) - \mathcal{K}(r) \mathcal{E}'(r) + r^2 \mathcal{K}(r) \mathcal{K}'(r) \right\}, \quad (3.4) \]
\[ \lim_{K \to 1} \frac{1 - g(r, K)}{K - 1} = \frac{1}{r'^2 \mathcal{K}^2(r)} \lim_{K \to 1} \frac{r'^2 \mathcal{K}^2(r) - s'^2 \mathcal{K}^2(s)}{K - 1} = \frac{4}{\pi} \mathcal{K}'(r) \left[ \mathcal{K}(r) - \mathcal{E}(r) \right], \quad (3.5) \]
and
\[ \lim_{K \to 1} \frac{[1 - g(r, K)]^2}{(K - 1)} = 0. \quad (3.6) \]

Let \( f_2(r, K) = \frac{\pi}{4} r'^2 f_1(r, K)/(K - 1) \) for \( K \in (1, \infty) \) and \( f_2(r, 1) = \lim_{K \to 1} f_2(r, K) \). then, by (3.4)–(3.6), it follows that
\[ f_2(r, 1) = r'^2 \mathcal{K}'(r) [\mathcal{K}(r) - \mathcal{E}(r)] - \mathcal{E}'(r) [\mathcal{E}(r) - r'^2 \mathcal{K}(r)] \]
and hence, \( f_2(1, 1) = -\pi/2 \). Hence \( f \) cannot be convex on \( (0, 1) \), when \( K \) is close to 1.

On the other hand, it is clear that, when \( K = 2 \), \( f(r) = 2/(1 + r) \) is convex on \( (0, 1) \).

3.7. **Proof of Theorem 1.15.** Set \( x^2 = 2t/(1 + 2t) \), \( r^2 = t/(1 + t) \) and \( y^2 = \sqrt{2t}/(1 + \sqrt{2t}) \), with \( x, r, y > 0 \). Then \( r = x/\sqrt{2 - x^2} \), \( y^2 = x/(x + x') \),
\[ \frac{dr}{dx} = \frac{rr'}{xx'^2}, \quad \frac{dy}{dx} = \frac{yy'}{2xx'^2} \quad (3.8) \]
and
\[ F(t) = F_1(x) = \frac{\mu(x) + \mu(y)}{\mu(r)} \]
It is sufficient to prove that $F_1$ is strictly increasing from $(0, 1)$ onto $(3/2, 3)$.

By differentiation,

$$\frac{4}{\pi} \frac{x(x'K(r)K(y)K(y))^{2}}{K^{2}(x)+2K^{2}(y)}F_{1}'(x) = F_{2}(x)$$

by virtue of (2.3) and (3.8), where

$$F_{2}(x) = 2K(x)K(y)\frac{K'(x)K(y) + K(x)K'(y)}{K^{2}(x)+2K^{2}(y)} - K(r)K'(r).$$

Next, by Lemma 2.6,

$$2K(y)K'(y) > \pi \log 2 + K(x)K'(x),$$

so that

$$F_{2}(x) > F_{3}(x) + \frac{K^{2}(x)}{K^{2}(x)+2K^{2}(y)}\pi \log 2,$$

where $F_{3}(x) = K(x)K'(x) - K(r)K'(r)$.

It is easy to verify that $y < \sqrt{x}$. Hence

$$\frac{K(y)}{K(x)} < \frac{K(\sqrt{x})}{K(x)} < \sqrt[4]{1+x} < \sqrt[4]{2},$$

by [QV, Theorem 1.5] or [AVV11, Theorem 3.11]. On the other hand,

$$F_{3}(x) > -\frac{\pi}{4} \log 2, \quad \text{for} \quad 0 < x < 1,$$

by Lemma 2.13.

From (3.11) and (3.12), it follows from (3.10) that

$$F_{2}(x) > -\frac{\pi}{4} \log 2 + \frac{\pi \log 2}{1 + 2[K(y)/K(x)]^2}$$

$$> -\frac{\pi}{4} \log 2 + \frac{\pi \log 2}{1 + 2\sqrt{2}} = \frac{(3 - 2\sqrt{2})\pi}{4(1 + 2\sqrt{2})} \log 2 > 0,$$

for $x \in (0, 1)$. Hence, $F_{1}'(x) > 0$ for $x \in (0, 1)$ by (3.9), yielding the monotonicity of $F_1$.

Finally, we have the limiting values:

$$F(0^+) = F_1(0^+) = \lim_{x \to 0} \frac{\log(4/x) + \log(4/y)}{\log(4/r)} = \frac{3}{2},$$
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\[ F(\infty) = F_1(1^-) = \lim_{x \to 1^-} \frac{\mathcal{K}(r)}{\mathcal{K}(y)} \left[ 1 + \frac{\mathcal{K}(y)}{\mathcal{K}(x)} \right] \]

\[ = \lim_{x \to 1^-} \frac{\log(4/r')}{\log(4/y')} \left[ 1 + \frac{\log(4/y')}{\log(4/x')} \right] = 3. \]

This completes the proof. \( \square \)

3.13. Proof of [Theorem 1.17]. (1) We need only prove the lower bound in (1.18). For \( n \geq 3 \), this was proved in [AVV8, Theorem 1.8 (4)], while for \( n = 2 \), it follows from [2.17].

(2) From the second inequality in [AVV8, Theorem 1.8 (1)] and the monotonicity of \( \tau_n(t) \), it follows that

\[ \tau_n(pt) + \tau_n((pt)^{1/p}) \leq (1 + p^{n-1}) \tau_n(pt) \leq (1 + p^{n-1}) \tau_n(t). \]

Next, it follows from the first inequality in [AVV8, Theorem 1.8 (1)] and (1.18) that

\[ \tau_n(pt) + \tau_n((pt)^{1/p}) \geq \left( 1 + \frac{1}{p} \right) \tau_n(pt) \geq \left( 1 + \frac{1}{p} \right) C_n(p) \tau_n(t). \]

For (3), we note that \( t \geq (pt)^{1/p} \) for \( t \geq p^{1/(p-1)} \). Hence, by (1.18),

\[ \tau_n(pt) + \tau_n((pt)^{1/p}) \geq \tau_n(pt) + \tau_n(t) \geq (1 + C_n(p)) \tau_n(t). \]

\( \square \)

3.14. Proof of [Theorem 1.23]. Let \( s = \varphi_K(r) \). Making use of (2.5), we get

\[ \frac{K^2g'(K)}{g(K)} = \frac{2}{\pi} s' K^2(s) \frac{K'(r)}{K(r)} + \log r - (1 + r') \log(1 + r'), \quad (3.15) \]

by logarithmic differentiation. Since \( s' K^2(s) \) is strictly decreasing in \( K \) on \([1, \infty) \) [AVV10, Theorem 2.2 (3)], it follows from (3.15) and Lemma 2.20 that

\[ K^2g'(K)/g(K) < m(r) + \log r - (1 + r') \log(1 + r') < 0. \]

\( \square \)

3.16. Proof of [Theorem 1.25]. The result (a) follows from the inequality (1.24) and the quasiconformal Schwarz lemma, namely [LV],

\[ |f(z)| \leq \varphi_K(|z|). \quad (3.17) \]
The result (b) follows from [AVV6, Lemma 4.6] and (1.24), while part (c) follows from part (b).

\[\square\]

**Remark 3.18.** The inequality (1.24) enables one to improve some known distortion results for quasiconformal mappings. *Theorem 1.25* is only one such example.

**Conjecture 3.19** Our computational work supports the validity of the following conjecture:

For each \( p > 1 \), the function \( f(t) = [\tau(pt) + \tau((pt)^{1/p})]/\tau(t) \) is strictly increasing from \((0, \infty)\) onto \((1 + 1/p, p + 1)\).

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